Robust Equilibria in Tournaments with Externalities

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Abstract

In a tournament, agents form coalitions with externalities, and the coalition with largest power wins the tournament. We introduce a new solution concept for the tournaments, called *no threat equilibrium (NTE)*. NTE is a partition of the agents, where the prudent agents have no incentive to deviate. Whenever a group of agents can gain by forming a coalition, a coalition with larger power can be formed by some other agents to defeat them as reaction to the deviation. For any power function and any preferences, the NTE exists if and only if the set of feasible coalitions is a Helly family. We also study the *core* of a tournament, and find that NTE expands the class of feasible coalitions, in which the core, might not exist. Furthermore, the applications of NTE, and farsighted core are discussed.

Keywords: Coalition Formation, Robust Equilibria, Tournament, Externalities

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1 Introduction

Coalitions are often formed around multiple issues at any scale of society, from neighborhood issues to international conflicts. Two aspects constantly affect the coalition formation in the conflicts. On the one hand, weaker parties can form coalition with others to increase their power and chances to win in conflicts, so that their interests are defended. Therefore, there is a need to develop a model of coalition formation that incorporates *power*. On the other hand, people prefer to associate with others in terms of similar goal and characteristics. Two parties belonging to the same political spectrum might benefit positively if the other wins the election. A community may dislike to associate with another from different background. Therefore, there is a need to analyze how preferences of the coalitions affect the result of coalition formation, i.e. how coalitions are influenced by *externalities* from other agents.

Contrast to the widely studied coalition formation game in which each player's payoff depends only on the members of his coalition, we introduce a model (henceforth called a *tournament*) in which agents form coalitions with externalities. The coalition with the greatest power wins the tournament. Agents belonging to a winning coalition receive the benefits, whereas losing agents receive nothing. In our model, the power of a coalition is represented by an arbitrary function that is non-decreasing with respect to the inclusion relation. The preferences of an agent are represented by arbitrary preferences over the coalitions to which the agent belongs. We focus on the structure of coalitions can be formed, which guarantees the existence of an equilibrium that is robust to power of coalitions and preferences of agents.

The core is widely used to study the stable structure of coalition formation problems. In the core, there is no profitable deviation for any group of agents by forming a coalition, assuming that other agents have no reactions to the deviation. However, in tournament with externalities, the deviations of coalitions affect other agents, which provide incentive for agents to reorganize and form new coalitions in order to win the tournament. It is reasonable for prudent agents to consider potential reactions of other agents after they deviate to form a coalition. The prudent agents will not deviate if their coalition is going to lose in tournament for the reactions. To address these issues, we introduce a new solution concept, named no-threat equilibrium (NTE). In no-threat equilibrium, any coalitions of prudent agents have no incentive to deviate, since the deviation is going to be defeated by a new coalition formed by other agents as reactions. In other words, no coalition of prudent agents can pose a threat to deviate from the equilibrium partition.

In order to illustrate our model and results, consider an example with a group of three agents $\{1,2,3\}$. The feasible coalitions are $\{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$. Assume the power of coalitions ranking from high to low is $\{1,2\}$, $\{2,3\}$, $\{1\}$, $\{2\}$, $\{3\}$, and the preferences satisfy $\{1\} \succ_1 \{1,2\}$, $\{1,2\} \succ_2 \{2\} \succ_2 \{2,3\}$, $\{2,3\} \succ_3 \{3\}$. There are 3 possible partitions: $\{\{1\},\{2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1\},\{2\},\{3\}\}$. For partition $\{\{1\},\{2,3\}\}$, the coalition $\{2,3\}$ wins, but agents 1 and 2 have incentive to deviate and form a coalition, both will be better off. For partition $\{\{1\},\{2\},\{3\}\}$, the coalition $\{1\}$ wins, agents 2 and 3 have incentive to form a coalition and win the tournament. For partition $\{\{1,2\},\{3\}\}$, agent 1 has incentive to form a coalition alone. Thus, the core does not exist. However, partition $\{\{1,2\},\{3\}\}$ is NTE, if agent 1 deviate to coalition $\{1\}$, he can expect the reactions of agent 2 and 3 to form a coalition $\{2,3\}$ and lose, so agent 1 can not pose a threat to deviate from the partition. From this example, NTE exists while the core does not exist, and NTE is a good prediction of coalition formation for prudent agents.

First, we study the NTE of tournaments with externalities. Our main results provide necessary and sufficient conditions for the existence of an NTE. Indeed, Theorem 1 shows that the NTE exists for any power function and any preferences if and only if the set of feasible coalitions is a Helly family. Moreover the NTE exists for any power function and any preferences if and only if any set of potentially winning coalitions is a subset of connected coalitions of a network without cycles (Proposition 1).

Then we study the core in tournaments, and our results show that the existence of core can be guaranteed if and only if two non-disjointed coalitions satisfy inclusion relation (Proposition 2), which is highly restrictive on the feasible coalitions. Furthermore, we discuss the applications of power matching problem and voting problem. In power matching problem, NTE always exists (Proposition 3), but the core might not.

Finally, we study the farsighted core of the tournaments, which is stronger than core. Thus, the class of feasible coalitions for existence of farsighted core is more restrictive than the core. The farsighted core exists for any preferences and any power function, if and only if the feasible coalitions have level of structure with less than 3 levels (Corollary 4).

1.1 Related Literature

A large swathe of literature studies equilibria in coalition formation with externalities, with many of them resembling the core. For instance, Bogomolnaia and Jackson [5] study different stability notions and characterize necessary and sufficient conditions for their existence. This notion has been extended to a variety of settings, for instance in Papai [23], Ehlers [13], Bloch and Dutta [4], Chatterjee et al. [7], Pycia [25], Romero-Medina [29], Banerjee et al. [2]. Several of these works include equilibria that work in dynamic settings as well, for instance Greenberg [15], Chwe [8], Bloch [3], Xue [31], Arnold and Schwalbe [1], Diamantoudi and Xue [10], Ray and Vohra [27, 28], Iñarra et al. [17]. Sasaki and Toda[30] study the existence and optimality of stable matching in two-sided matching markets with externalities. Pycia and Yenmez[26] study two-sided matching problem with externalities, extend the substitutes condition for the existence of stable matching. Braitt and Torres-Martínez[11] study the roles of prudence and social connectedness in asymptotic stability when the externalities and preferences are random. Unfortunately, such literature does not study the role that power plays in the formation of coalitions.

A recent paper does focus on the issue of power. Piccione and Razin [24] study how the power relations over coalitions of agents determine the overall ranking of society, and characterize the stable social order. Similarly, Jordan [21] outlines the core and the stable set in a class of coalitional games called "pillage games", in which the wealth is allocated amongst the finite agents. A reallocation of wealth among the agents is only possible by using force. A power function, which monotonically increases in terms of membership and members' wealth, regulates the ability of the agents to use force. The coalition is then able to appropriate the wealth of other less powerful coalitions. Although newer coalition formation models are dynamic in nature, most of them are limited to a static distribution of power among the agents. We plug this gap by exploring the possibility that agents are able to deviate from a currently formed coalition. Jandoc and Juarez[18] study a model in which agents are endowed with power and characterize the formation of coalitions when power accumulates. The coalition formation is highly dependent on the power accumulation rule. Jandoc and Juarez[20] study dynamic coalition formation when agents disagree on the sharing rule. It characterizes the structure of power that guarantees the existence of stable coalitions, even if they disagree on how to share the resource. Jandoc and Juarez[19] test the self-enforcing coalition formation equilibrium, and find that the equilibrium is a good predictor for coalitions formed over time.

There is a stream of literature studying the stable solutions for models of coalition formation without externalities. Demuynck et al.[9] study the myopic stable set in a general class of environments, which unifies core in coalition function form game, stable matching, and pairwise stable network in models of network formation, etc. Herings et al.[16] study matching problem with myopic and farsighted players. Kondratev and Mazalov[22] study the solution for tournament without externalities from perspective of cooperative game theory. Despite the abundance of coalition formation models, little work has been done on studying the robustness of equilibria, especially when the power of the agents or their preferences may change. Our work is the first to introduce and characterize the structure upon which coalitions can form and guarantee the existence of equilibria, regardless of whether the power and preferences of agents change.

2 The model

We study coalition formation models for a fixed group of agents $N = \{1, \dots, n\}$ who are endowed with power, which is an arbitrary function among coalitions.

Definition 1. *A power function as a mapping is indicated as* $\pi : 2^N \to \mathbb{R}_+$, *such that:*

- $\pi(\emptyset) = 0$.
- *If* $S \subseteq T$, *then* $\pi(S) \leq \pi(T)$.
- *If* $S \cap T = \emptyset$, $\pi(S) \neq 0$ *and* $\pi(T) \neq 0$, *then* $\pi(S) \neq \pi(T)$.

This function represents a ranking of power over the set of coalitions. $\pi(S) > \pi(T)$ means that coalition S is more powerful than coalition T. Moreover, two disjointed coalitions cannot have the same positive power, which breaks ties in competitive situations between different groups of players.

Some coalitions are able to form, while others cannot do so. This might represent, for instance, a physical constraint.

Definition 2. *The set of feasible coalitions* $\mathbb{F}$ *is a collection of subsets* $\mathbb{F} \subseteq 2^N$, *such that if* $S, T \in \mathbb{F}$, *there is a partition of* $S \setminus T$ *composed of sets in* $\mathbb{F}$.

We interpret $\mathbb{F}$ as the set of coalitions that can be formed. If coalition S is formed but players in $S \cap T$ decide to form a coalition T, then the remaining players in $S \setminus T$ can be rearranged into feasible coalitions.

Example 1.

- *If $\mathbb{F} = 2^N$, then every group of agents is a feasible coalition.*
- *If $\mathbb{F} = \{\{1\}, \{2\}, \ldots, \{n-1\}, \{n\}\}$, then no group of two or more agents can be formed.*
- *Let $\mathbb{F}$ be the set of coalitions containing one or two agents.*
- *Fix a group of agents T, if $\mathbb{F} \subset \{S \mid T \subseteq S\}$, in which case any coalition needs the group T to be formed.*

Network analysis would be helpful in finding the types of feasible coalitions. Given a network, we will show how to generate a set of feasible coalitions, a central idea behind which is our notion of "connectedness."

Definition 3. *Consider a network H made up of N agents. We say a coalition S is* **connected** *if the sub-network restricted to agents in S has a single component.*

Denote by $C(H)$ the set of all connected coalitions. As we shall see below, the set $\mathbb{F}$ of feasible coalitions will be related to the set $C(H)$ for some networks.

We can interpret H as the network of friendships, i.e. two agents are friends if there exists a link in H that connects them. Alternatively, H might be interpreted as the location of roads or the political affiliation of agents.

Example 2.

- *If M is a line, then $\mathbb{F}$ contains all the segments that are subsets of M. We can interpret M as the set of agents with an affiliation from leftist to rightist. Feasible coalitions are those that contain only agents with consecutive characteristics.*

2.1 Tournament

A player i has a strict preference $\succ_i$ over feasible coalitions that contain himself and the empty set. That is, the domain of the agent's preference i is $\mathbb{S}_i = \{S \in \mathbb{F} \mid i \in S\} \cup \{\emptyset\}$, such that $S \succ_i \emptyset$ for all $S \in \mathbb{S}_i$. A tournament in this setting is a game in which agents form coalitions, and the coalition with the largest amount of power wins the tournament.

Given N players, let the partitions of set N be denoted by Π.

Definition 4. *A tournament is a function $G : \Pi \to 2^N$, such that for all $P \in \Pi$:*

a. $G(P) \in P$.

b. $\pi(G(P)) \geq \pi(S)$, $\forall S \in P$.

The outcome of a tournament for agent i at partition P is:

$$G_i(P) = \begin{cases} G(P) & \text{if} \quad i \in G(P) \\ \emptyset & \text{if} \quad i \notin G(P) \end{cases}$$

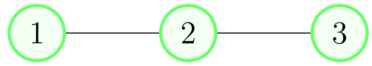

Figure 1: Network of three agents in a line

That is, the function G gives the winning coalition for every partition P. This coalition has the greatest amount of power from P.

2.1.1 No-threat Equilibrium

Definition 5. *The set of winning coalitions is:*

$$W = \{S \in \mathbb{F} \mid \pi(S) > \pi(T), \forall T \subseteq N \setminus S,\ T \in \mathbb{F}\}.$$

Definition 6. *Let $P \in \Pi$ be a partition and $T \in \mathbb{F}$ be a feasible coalition. The set*

$$[P \setminus T] = \{\bar{P} \in \Pi(N \setminus T) \mid \textit{if } S \in P \textit{ and } S \setminus T \in \mathbb{F} \textit{ then } S \setminus T \in \bar{P}\}$$

represents a class of maximal partitions.

Definition 7. *We say that a coalition $T \in \mathbb{F}$ defeats a partition $P \in \Pi$ if there is $\bar{P} \in [P \setminus T]$, such that*

$$G_i(\bar{P}, T) \succ_i G_i(P),\ \forall i \in T,$$

where $(\bar{P}, T)$ denotes the partition $\bar{P} \cup \{T\}$ of N.

In the previous definition, note that coalition T must be a winning coalition in $(\bar{P}, T)$, in order to defeat partition P. Thus, it will cause no confusion if we say that the winning coalition in a partition defeats the winning coalition in another partition.

Equilibrium-winning coalitions need not necessarily be stable in relation to the deviations of a group of players. Thus, we employ the stability notion, referred to as the "no-threat equilibrium" (NTE).

Definition 8. *Let G be a tournament. A partition P^* is an NTE if whenever there is a coalition $T \in \mathbb{F}$ and $\bar{P} \in [P^* \setminus T]$ such that $(\bar{P}, T)$ defeats P^*, in which case there is a coalition $V \subseteq N \setminus T$ and $\tilde{P} \in [(\bar{P}, T) \setminus V]$, such that $(\tilde{P}, V)$ defeats $(\bar{P}, T)$.*

We now show examples of networks that give rise to a set of feasible coalitions where the NTE may or may not exist.

Example 3. *Consider the network in Figure 1, which gives the set of connected coalitions*

$$C(H) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Suppose that the feasible set is equal to $C(H)$, and let the power function be

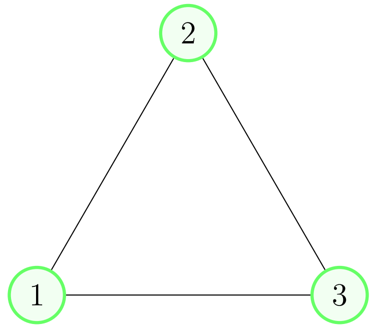

Figure 2: Network with three agents in a cycle.

$$\pi(\{2\}) < \pi(\{1\}) < \pi(\{3\}) < \pi(\{1,2\}) < \pi(\{2,3\}) < \pi(\{1,2,3\})$$

Assume that preferences are as follows:

$\{1\} \succ_1 \{1,2\} \succ_1 \{1,2,3\}$
$\{2\} \succ_2 \{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\}$
$\{3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\}$

In this case, {1,2} is the NTE. If player 1 deviates to form, say a singleton coalition $\{1\}$, then another feasible coalition—in this case $\{2,3\}$—can be formed and win over $\{1\}$.

Example 4. *Now consider the network in Figure 2 that generates the set of connected coalitions:*

$$C(H) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}.$$

Suppose that $\mathbb{F} = C(H)$, and let π the power function be

$$\pi(\{2\}) < \pi(\{1\}) < \pi(\{3\}) < \pi(\{1,2\}) < \pi(\{2,3\}) < \pi(\{1,3\}) < \pi(\{1,2,3\})$$

$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\}$
$\{2,3\} \succ_2 \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\}$
$\{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}$

In this case, the NTE does not exist, since this network generates a cycle. Thus, player 2 will deviate from $\{1,2\}$ and $\{2,3\}$ can be formed, while coalition $\{2,3\}$ defeats $\{1,2\}$ but $\{1\}$ cannot defeat $\{1,2\}$. However, player 3 would like to deviate and $\{1,3\}$ can be formed, and so coalition $\{1,3\}$ defeats $\{2,3\}$ but $\{2\}$ cannot defeat $\{1,3\}$. Nonetheless, player 1 would like to form $\{1,2\}$, in which case coalition $\{1,2\}$ defeats $\{1,3\}$ but $\{3\}$ cannot defeat $\{1,2\}$, and so on.

3 Main Theorem

The main theorem in this paper illustrates the set of restrictions on $\mathbb{F}$ that guarantee the existence of the NTE for all preferences and all power functions. Moreover, if $\mathbb{F}$ are rich enough, the set of feasible coalitions will come from a set of connected coalitions in a network without cycles.

Note that any set of winning coalitions is pairwise non-disjoint. This is, let W be a set of winning coalitions and $S, T \in W$, then $S \cap T \neq \emptyset$ [1]. Thus, the following definition establishes a necessary condition for a set that has the opportunity to be a winning condition.

Let $[k]$ be the set $\{1, 2, ..., k\}$ for every $k \in \mathbb{N}$.

Definition 9. *We say that a family $\{S_1, S_2, \ldots, S_k\} \subseteq \mathbb{F}$ is composed of* ***potentially winning*** *coalitions if $S_i \cap S_j \neq \emptyset$ for all $i, j \in [k]$.*

The following stronger condition will guarantee us the existence of at least one NTE.

Definition 10. *A collection of feasible coalitions $\mathbb{F}$ is* ***a Helly family*** *if every family $\{S_1, S_2, \ldots, S_k\} \subset \mathbb{F}$ composed of potentially winning coalitions satisfies the notion that $\bigcap_{i \in [k]} S_i \neq \emptyset$.*

A family of sets (or coalitions) is equivalent to a *hypergraph*, if each set is a hyperedge. The Helly property has been widely studied, for instance, by the hypergraph theory [6, 12] which shows several characterizations for hypergraphs with this property. The feasible coalitions $\mathbb{F}$ is Helly family, if every family composed of potentially winning coalitions have at least one common agent.[2]

Theorem 1. *Given a set of feasible coalitions $\mathbb{F}$, an NTE exists for all preferences defined on $\mathbb{F}$ and all power functions, if and only if $\mathbb{F}$ is a Helly family.*

Proposition 1. *Let $\mathbb{F}$ be a set of feasible coalitions. An NTE exists for all preferences defined on $\mathbb{F}$ and all power functions, if and only if, for every set $S \subset \mathbb{F}$ that is composed of* potentially winning *coalitions, there exists a network without cycles H, such that $S \subseteq C(H)$.*

In the following example, there is a network H with a cycle such that $\mathbb{F} \subseteq C(H)$. Every set S composed of potentially winning coalitions belongs to connected coalitions of a line. Moreover, the set of feasible coalitions $\mathbb{F}$ is a Helly family, and an NTE exists for all preferences defined on $\mathbb{F}$ and all power functions.

Example 5. *Let $\mathbb{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ be the set of feasible coalitions. Now consider the network in Figure 3.*

Each edge of the square is a feasible coalition, but $\mathbb{F}$ is not composed of potentially winning *coalitions. In fact, if an edge belongs to a family composed of potentially winning coalitions, the opposite edge does not belong to that family.*

[1] This is proven in Step 1 of the proof of Theorem 1.

[2] See Bollobás[6] for definition of Helly family. For a family to be H_2-family, if any 2 coalitions have an agent in common, then there is an agent contained in all the coalitions. For a H_1-family, there exists an agent contained in all the coalitions. The feasible coalitions $\mathbb{F}$ is H_2-family, and every family composed of potentially winning coalitions is H_1-family.

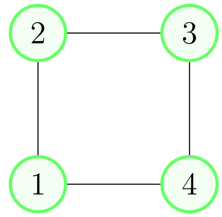

Figure 3: Network with four agents in a square.

For example, the edge $\{1,2\}$ only belongs to four families composed of potentially winning coalitions:

$$S_1 = \{\{1,2\},\{2,3\}\},\ S_2 = \{\{2\},\{1,2\},\{2,3\}\},$$
$$S_3 = \{\{1,2\},\{4,1\}\} \text{ and } S_4 = \{\{1\},\{1,2\},\{4,1\}\}.$$

All families S_1, S_2, S_3 and S_4 have agents in common.

This is analogous for families composed of potentially winning coalitions to which the other edges belong, every family composed of potentially winning coalitions have agents in common. Therefore $\mathbb{F}$ is a Helly family, an NTE exists for all preferences defined on $\mathbb{F}$ and all power functions.

4 The Core

In coalitional games, the core is an important solution concept. A partition is core stable if it cannot be improved by any coalition. In this section, we first define the core in tournaments with externalities. Then we characterize the necessary and sufficient condition for existence of core in tournaments. Finally, we reformulate the stable matching problem and compare the core and NTE.

Definition 11. *A partition $P \in \Pi$ is **stable** if no coalition defeats it. The **core** is the set of all stable partitions.*

If a partition P is not defeated by any coalition, then no coalition could improve by deviation, which is the idea of core. We characterize the conditions for existence of core in the following proposition.

Proposition 2. *Given a set of feasible coalitions $\mathbb{F}$, the core is not empty for all preferences defined on $\mathbb{F}$ and all power functions, if and only if for all pairs of coalitions $A, B \in \mathbb{F}$, such that $A \cap B \neq \emptyset$ then $A \subset B$ or $B \subset A$.*

Corollary 2. *If the core is not empty for all preferences defined on $\mathbb{F}$ and all power functions, then there exists a line H such that $\mathbb{F} \subset C(H)$.*

In proposition 2, the existence of core requires the set of feasible coalitions $\mathbb{F}$ could be defined on a line, the condition is stronger than the condition for existence of NTE in proposition 1, which requires the set of feasible coalitions $\mathbb{F}$ to be defined on a network without cycles H. In corollary 2, feasible coalitions $\mathbb{F}$ is a subset of connected coalitions on a line. In the following corollary, we show that if the core exists, then an NTE exists.

Corollary 3. *If the core is not empty for all preferences defined on $\mathbb{F}$ and all power functions, then an NTE exists.*

However, the existence of NTE is not sufficient for existence of the core. The counter example of result is discussed in the following power matching problem.

4.1 Power Matching Problem

We reformulate the stable matching problem presented by Gale and Shapley [14] in the following setting, and we add the possibility that each potential couple (or single) has some level of power in the matching process. The classical matching problem can be modeled with a null power function.

Consider the joint business ventures competitions, let L be a set of labor business and C a set of capital business (and $L \cap C = \emptyset$). Then, the set of business is $N = L \cup C$ and the set of feasible coalitions $\mathbb{F} = \{\{i\} : i \in N\} \cup \{\{l, c\} : l \in L, c \in C\}$ are the singletons of each business joined with every possible joint business ventures among labor business L and capital business C. The preferences of each business are described as preferences over feasible coalitions that contain himself, that is, $\{l, c_1\} \succ_l \{l, c_2\}$ means that l prefers c_1 over c_2, and so forth.

Example 6. *Let $L = \{1, 2\}$ the set of labor business, $C = \{3, 4\}$ the set of capital business and $N = \{1, 2, 3, 4\}$ be the set of all business. Then, let*

$$\mathbb{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

be the set of feasible coalitions. The power of every singleton is null, and the ranking of power for joint business is

$$\pi(\{1, 4\}) > \pi(\{1, 3\}) > \pi(\{2, 3\}) > \pi(\{2, 4\})$$

Preferences are defined as follows:

$$\{1, 3\} \succ_1 \{1, 4\} \succ_1 \{1\}$$
$$\{2, 3\} \succ_2 \{2, 4\} \succ_2 \{2\}$$
$$\{2, 3\} \succ_3 \{1, 3\} \succ_3 \{3\}$$
$$\{2, 4\} \succ_4 \{1, 4\} \succ_4 \{4\}$$

Thus, each business prefers to be in any joint business ventures rather than to be alone.

$\{\{1, 3\}, \{2, 4\}\}$ *is an NTE, because if labor business* 2 *and capital business* 3 *decide to form a coalition* $\{\{1\}, \{2, 3\}, \{4\}\}$*, then* 1 *and* 4 *will form a coalition* $\{1, 4\}$ *and defeat* $\{2, 3\}$ *in the partition* $\{\{1, 4\}, \{2, 3\}\}$.

Proposition 3. *The power matching problem has an NTE for all preferences and all power functions.*

Note that if a partition P is stable, then P is an NTE, albeit the converse result does not hold. Moreover, the following example shows that the power matching problem in Example 6 is a situation in which there is an NTE but the core is empty.

Example 7. *Recall the situation in example 6. This tournament has an empty core. Indeed, it is not difficult to see that no matter what partition is formed, there is always a coalition that can defeat it.*
For instance, if the partition $\{\{1,3\},\{2,4\}\}$ *were formed, then coalition* $\{1,3\}$ *would win the tournament. Nevertheless, this partition is not stable, since agents* 2 *and* 3 *can defeat by deviating:* $\{\{1\},\{2,3\},\{4\}\}$.

The set of feasible coalitions $\mathbb{F}$ of power matching problem is connected coalitions of a bipartite network with at most 2 players. There are links between players in L and C, but the players are not connected within L or C. $\mathbb{F}$ is composed of one player or two players connected. There is no cycle in bipartite network, and the feasible coalitions are not connected coalitions of a line.

4.2 Voting Problem

Consider a voting problem among different parties, let A, B be the set of candidates from party A and B. The set of all candidates is $A \cup B$. There are level structure of coalitions. The first level of coalitions in party A are $\{A_1, \ldots, A_k\}$, the set of these coalitions is a partition of A. The second level of coalitions constitute a partition of the coalitions in first level, the set of coalitions $\{A_{i,1}, \ldots, A_{i,k_i}\}$ is a partition of A_i, for any $i \in [k]$. More levels of coalitions could be defined in this way.

Example 8. *Let* $A = \{1,2,3\}$ *be the set of agents of party* A, $B = \{4,5\}$ *the set of agents of party* B *and* $N = \{1,2,3,4,5\}$ *the set of all agents. Then, let*

$$\mathbb{F} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2,3\},\{1,2\},\{4,5\}\}$$

be the set of feasible coalitions.

Assume the power of singleton are: $\pi(\{1\}) = 1.1$, $\pi(\{2\}) = 0.9$, $\pi(\{3\}) = 0.7$, $\pi(\{4\}) = 1$, $\pi(\{5\}) = 0.8$. *Assume the power of coalition of more than one agents is the aggregate power of individuals in the coalition. Thus,* $\pi(\{1,2,3\}) > \pi(\{1,2\}) > \pi(\{4,5\}) > \pi(\{1\})$.

Preferences are defined as follows:

$$\begin{aligned}
\{1,2\} \succ_1 \{1,2,3\} &\succ_1 \{1\} \\
\{1,2,3\} \succ_2 \{1,2\} &\succ_2 \{2\} \\
\{1,2,3\} &\succ_3 \{3\} \\
\{4,5\} &\succ_4 \{4\} \\
\{4,5\} &\succ_5 \{5\}
\end{aligned}$$

$\{\{1,2\},\{3\},\{4,5\}\}$ is in the core. $\{1,2,3\}$ will not be formed because agent 1 prefer $\{1,2\}$, and 1 will not deviate to $\{1\}$ and lose the tournament.

Moreover, $\{\{1,2,3\},\{4,5\}\}$ is also in the core.

The structure of feasible coalition in example 9 has levels, $A_1 = A$, and $A_{1,1} = \{1,2\}$, $A_{1,2} = \{3\}$. In the level structure of coalitions, for any coalition $C, D \in \mathbb{F}$ with $C \cap D \neq \emptyset$, there is $C \subset D$ or $D \subset C$. Thus, the core exists in the voting problems.

4.3 Farsighted Core

Compare the tournaments of coalition with the proper simple game in Ray and Vohra [27], the core of a simple game is nonempty if and only if the collegium $S^* = \bigcap_{S \in W} S$ is nonempty. W is the set of all winning coalitions in the simple game.

Our results are different with simple game, and we focus on the structure of feasible sets rather than the set of winning coalitions. In the following, we define the defeat in chain of coalitions and farsighted stable.

Definition 12. *The chain of coalitions $S^1, \ldots, S^k$ that defeats a partition P, if $G_j(P^k) \succ_j G_j(P^{i-1})$, for any agent $j \in S^i$, $i = 1, \ldots, k$. The partitions $P^0, P^1, \ldots, P^k$ satisfy $P^0 = P$, $P^i = (S^i, [P^{i-1} \setminus S^i])$ for $i = 1, \ldots, k$.*

Definition 13. *A partition $P \in \Pi$ is* ***farsighted stable*** *if there is not chain of coalitions that defeats it. The* ***farsighted core*** *is the set of all farsighted stable partitions.*

If a partition P is farsighted stable, then no coalition could improve by a chain of deviations. To illustrate the results of farsighted stable, we define the level structure of feasible coalitions. The feasible coalitions $\mathbb{F}$ have level structure if any coalitions $A, B \in \mathbb{F}$, there is $A \subset B$ or $B \subset A$. $\mathbb{F}$ have level structure with k levels if $k = \max_{i \in N} |\{S | i \in S\}|$, which is the largest number of coalitions including any agent i.

Corollary 4. *Given a set of feasible coalitions $\mathbb{F}$, the farsighted core is not empty for all preferences defined on $\mathbb{F}$ and all power functions, if and only if the set feasible coalitions have level structure with less or equal than 2 levels.*

Corollary 4 shows there exists a partition that is farsighted stable if and only if the feasible coalition $\mathbb{F}$ have level structure with 1 level or 2 levels. The following example illustrates the intuition that farsighted stable partition does not exist for a set of feasible coalitions with level structure with 3 levels.

Example 9. *Let $N = \{1,2,3,4\}$ be the set of agents and $\mathbb{F} = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,2,3\}\}$ be the set of feasible coalitions. The coalitions including agent 1 are winning coalitions. Preferences are defined as follows:*

$$\{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\}$$
$$\{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2\}$$
$$\{1,2,3\} \succ_3 \{3\}$$

Note that the partitions $\{\{1,2\},\{3\},\{4\}\}$ and $\{\{1,2,3\},\{4\}\}$ are in the core.

Thus, the core is not empty set, but farsighted stable set does not exist.

If we start with partition $P^0 = \{\{1,2\},\{3\},\{4\}\}$, then agent 2 deviates as a coalition S^1 and the partition $P^1 = \{\{1\},\{2\},\{3\},\{4\}\}$ results. At this point, agents $1,2,3$ prefer to form a coalition S^2 and therefore, the partition $P^2 = \{\{1,2,3\},\{4\}\}$ is finally formed. The chain of coalitions S^1, S^2 defeats P^0, $\{\{1,2\},\{3\},\{4\}\}$ is not farsighted stable.

On the other hand, if we start with partition $\{\{1,2,3\},\{4\}\}$, then 1 deviates and the partition $\{\{1\},\{2\},\{3\},\{4\}\}$ is formed. But from here, agents $1,2$ prefer form a coalition and thus partition $\{\{1,2\},\{3\},\{4\}\}$ results.

Therefore, farsighted stable set is empty.

5 Conclusion

This paper investigates the robust equilibrium, "no-threat equilibrium", for the problem of tournaments with externalities. The no-threat equilibrium requires that any coalition of agents have no incentive to deviate for higher payoff and not being defeated by any feasible coalition formed by other agents. The condition is weaker than stability that requires agents have no incentive to deviate and achieve higher utility, since it excludes potential myopic deviations of agents that would be defeated by feasible coalition formed by other agents.

This paper is a start to study robust equilibrium in tournaments with externalities, we discover and characterize the set of feasible coalitions such that the no-threat equilibrium exists for any power function and preferences of agents. Then we characterized the condition of feasible coalitions for existence of core. Furthermore, the applications of no-threat equilibrium in power matching problem and voting problem are discussed. Finally, we discuss the farsighted stability in this problem, and characterize the feasible coalitions for existence of farsighted stable partition.

Proof

Proof of Theorem 1

Proof. ($\Leftarrow$) **Step 1:** Any collection of winning coalitions is a Helly family.

First, observe if S and T are winning coalitions, in which case $S \cap T \neq \emptyset$, because if $S \cap T = \emptyset$, then $T \subseteq N \setminus S$. However, by definition 5, if $S \in W$ and $T \subseteq N \setminus S$, then $T \notin W$.

Thus, if $S_1, S_2, \ldots, S_K \in W$ are winning coalitions, then $S_i \cap S_j \neq \emptyset$ for all $i, j \in \{1, 2, \ldots K\}$. Therefore, according to the hypotheses, we have $\bigcap_{k=1}^{K} S_k \neq \emptyset$.

Step 2: Let $V = \bigcap_{k=1}^{K} S_k$ and let $i \in V$. Next, let $S^* \in W$ such that $S^* \succ_i S$, $\forall S \in W \setminus \{S^*\}$. Let there be a partition $P = (S^*, [N \setminus S^*])$ where $[N \setminus S^*]$ is a partition of $N \setminus S^*$ composed of feasible coalitions. Assume that T deviates at partition P.

Case 1: $i \in T$. If $T \in W$, then i gets a lower pay-off. On the other hand, if $T \notin W$, there exists a coalition in $N \setminus T$ that can be formed and will defeat the coalition T.

Case 2: $i \notin T$. From our assumption, $T \notin W$, since $i \in V$ is part of all winning coalitions. Therefore, there exists a coalition in $N \setminus T$ that can be formed and will defeat the coalition T.

Therefore, S^* is an NTE. Note that every winning coalition at the top of players' preferences (restricted to W) is an NTE.

($\Rightarrow$) We will now prove the contrapositive version: Given a set of feasible coalition $\mathbb{F}$, if there is a collection $C = \{S_1, S_2, \ldots, S_K\} \subseteq \mathbb{F}$, such that $S_i \cap S_j \neq \emptyset$, $\forall i, j$ but $\bigcap_{k=1}^{K} S_k = \emptyset$, then there are preferences defined on $\mathbb{F}$ and a power function, such that an NTE does not exist.

Consider the power function π as follows:

$$\pi(S) = \begin{cases} 1 & \text{if} \quad S \in C \\ 0 & \text{if} \quad S \notin C \end{cases}$$

Clearly, $S_1, S_2, \ldots, S_K$ are the only winning coalitions for this power function.

Now, we will prove that there are preferences defined on $\mathbb{F}$, such that an NTE does not exist.

Let $i \in N$ be any agent. Due to $\bigcap_{k=1}^{K} S_k = \emptyset$, we find that i belongs in $r < K$ winning coalitions. Assume that i belongs to the coalitions $S_{i_1}, S_{i_2}, ..., S_{i_r}$, such that $i_1 < i_2 < ... < i_r$.

Consider the following strict preference over $\mathbb{F}$ for each agent $i \in N$:

- If $i \notin S_1 \cap S_K$, then
$$S_{i_1} \prec_i S_{i_2} \prec_i ... \prec_i S_{i_r}.$$
- If $i \in S_1 \cap S_K$, then
$$S_{i_r} \prec_i S_{i_1} \prec_i S_{i_2} \prec_i ... \prec_i S_{i_{r-1}}.$$

Observe that these preferences satisfy the following conditions:

$$S_1 \prec_j S_2, \quad \forall j \in S_1 \cap S_2$$
$$S_2 \prec_j S_3, \quad \forall j \in S_2 \cap S_3$$
$$\vdots$$
$$S_K \prec_j S_1, \quad \forall j \in S_K \cap S_1.$$

These properties in the preferences mean that the NTE does not exist, since this situation generates a cycle. Thus, for $r = 1, ..., K-1$, the group $S_r \cap S_{r+1}$ will deviate from S_r and S_{r+1} can be formed, following which coalition S_{r+1} defeats coalition S_r, albeit there are no winning coalitions in $N \setminus S_{r+1}$. Finally, group $S_K \cap S_1$ would like to deviate and S_1 can be formed, in which case coalition S_1 defeats coalition S_K and there are no winning coalitions in $N \setminus S_1$. Nevertheless, group $S_1 \cap S_2$ would like to form S_2, and so this coalition defeats coalition S_1 again, and so on.

□

Proof of Proposition 1

Proof. ($\Leftarrow$) Assume that $S = \{S_1, S_2, \ldots, S_K\}$ is composed of potentially winning coalitions and there is a network without cycle H such that $S \subseteq C(H)$.

First, we show that $\bigcap_{k=1}^{K} S_k \neq \emptyset$.

We prove by induction that $\bigcap_{k=1}^{K} S_k \neq \emptyset$. The base of induction is $n = 3$, and clearly we can arrange the players in a line H (see Figure 1), because this is the only possible network for three players if there are no cycles. There are essentially two cases in which S is composed of potentially winning coalitions and $S \subseteq C(H)$ [3]:

- $S = \{\{i\}, \{i, 2\}\}$ for $i = 1$ or 3. Thus, player i belongs to every potentially winning coalition.
- $S = \{\{1, 2\}, \{2, 3\}\}$. Thus, player 2 belongs to every potentially winning coalition. This case is analogous for $S = \{\{2\}, \{1, 2\}, \{2, 3\}\}$.

Let our induction hypothesis be that this is true for any network for n players. We will now prove this for $n + 1$ players. Let $\{S_1, S_2, \ldots, S_K\}$ be elements in the set of connected coalitions of the $n + 1$ network structure, such that $S_i \cap S_j \neq \emptyset$.

• If $S_1, S_2, \ldots, S_K$ only contains n players, then we are done by the induction hypothesis.

• Assume that all $n + 1$ players are in $S_1, S_2, \ldots, S_K$. Take an agent a with degree one in the network and take an agent b that is connected to a. If $a \in \bigcap_{k=1}^{K} S_k$, we are done.

• If $a \notin \bigcap_{k=1}^{K} S_k$, we know that $a \notin S_k$ for some k. We show that if $a \in S_i$, it must be that $b \in S_i$. To exemplify this point, it is noteworthy that if $a \notin S_k$, then for some player $x \neq a$ it must be that

[3]The proof is analogous when $\{1, 2, 3\} \in S$.

$x \in S_i \cap S_k$. Take the path $[x, a]$. Since $a \in S_i$ and $x \in S_i$, by connectedness it must be that players on the path $[x, a]$ should be a subset of S_i, i.e. $[x, a] \subseteq S_i$. Moreover, since $b \in [x, a]$, $b \in S_i$.

• Now take the sets $S_1 \setminus \{a\}, S_2 \setminus \{a\}, \ldots, S_K \setminus \{a\}$. We know $(S_i \setminus \{a\}) \cap (S_j \setminus \{a\}) \neq \emptyset$, because $S_i \cap S_j \neq \emptyset$, and if $a \in S_i$, then $b \in S_i$. Since these sets only contain n players and they are subsets of network H without the link [a,b] (which is a network without cycles), then according to the induction hypothesis $\bigcap(S_i \setminus \{a\}) \neq \emptyset$ implies $\bigcap(S_i) \neq \emptyset$.

Therefore, in line with Theorem 1, we can conclude that an NTE equilibrium exists for all preferences defined on $\mathbb{F}$ and all power functions.

($\Rightarrow$) Suppose that $S = \{S_1, S_2, \ldots, S_K\}$ is composed of potentially winning coalitions and an NTE exists for all preferences defined on $\mathbb{F}$ and all power functions.

We prove by induction that there is a network H without cycles, such that $S \subseteq C(H)$. The base of the induction is $n = 3$. As we show in the "only if" part, three players in all families of potentially winning coalitions can be arranged in one line.

The induction hypothesis is that this is true for any network for n players. We then prove this for $n+1$ players. Let $S = \{S_1, S_2, \ldots, S_K\}$ be a family composed of potentially winning coalitions with $n+1$ players. With Theorem 1, we have $\bigcap_i S_i \neq \emptyset$.

If $K = 1$, trivially the network of $n+1$ players in a line works.

If $K > 1$, there exists a coalition $S_r \neq \bigcap_i S_i$, such that there is a player $a \notin \bigcap_{k=1}^{K} S_k$. We take the coalitions $S_1 \setminus \{a\}, S_2 \setminus \{a\}, \ldots, S_K \setminus \{a\}$, whereby $\bigcap(S_i \setminus \{a\}) \neq \emptyset$. Then, with the induction hypothesis, there is a network G without cycles on $N \setminus \{a\}$, such that $\{S_1 \setminus \{a\}, S_2 \setminus \{a\}, \ldots, S_K \setminus \{a\}\} \subseteq C(G)$.

For some $b \in \bigcap_i (S_i \setminus \{a\})$, we define the network $H = G \cup [b, a]$ by adding the node a and linking $[b, a]$ to network G. Clearly, H has no cycles. Moreover, due to $b \in \bigcap_i (S_i \setminus \{a\})$ and $[b, a] \in H$, every $S_k \in S$ belongs $C(H)$ for $k = 1, \ldots, K$.

□

Proof of Proposition 2

Proof. ($\Leftarrow$) Let $W = \{S_1, ..., S_m\}$ be the set of winning coalitions. By definition, we find that

$S_i \cap S_j \neq \emptyset$ for all $i, j = 1, ..., m$. Thus, with hypothesis, $S_i \subset S_j$ or $S_j \subset S_i$. Without loss of generality, suppose that $S_1 \subset S_2 \subset \cdots \subset S_m$.

Let $i \in S_1$ and let $S^* \in W$, such that $S^* \succ_i S$, $\forall S \in W \setminus \{S^*\}$. Note that i belongs to every winning coalition.

There is a partition $P = (S^*, [N \setminus S^*])$, where $[N \setminus S^*]$ is a partition of $N \setminus S^*$ composed of feasible coalitions. Assume $\bar{P} \in \Pi(N \setminus S^*)$ is the maximal partition of $N \setminus S^*$, which means for any partition $\tilde{P} \in \Pi(N \setminus S^*)$, for any coalition $T_1 \in \tilde{P}$, there exists $T_2 \in \bar{P}$, such that $T_1 \subseteq T_2$. In the following, we prove the partition $\bar{P} \cup \{S^*\}$ is stable.

Suppose that T defeats the partition P.

Case 1: $i \in T$. If $T \in W$, then i does not prefer T over S^*.

If $T \notin W$, and $i \in T \cap S^*$. $T \cap S^* \neq \emptyset$, then $T \subset S^*$ and there exists a coalition $R \subseteq N \setminus T$, such that $\pi(R) > \pi(T)$. If $R \cap S^* \neq \emptyset$, there is $R \subseteq S^* \setminus T$, $\pi(S^* \setminus T) \geq \pi(R) > \pi(T)$, then T loses the tournament, and T does not defeat partition P. If $R \cap S^* = \emptyset$, $R \subseteq N \setminus S^*$, $\bar{P}$ is the maximal partition of $N \setminus S^*$, there is a set $T_1 \in \bar{P}$ such that $R \subseteq T_1$, $\pi(T_1) \geq \pi(R) > \pi(T)$, T loses the tournament.

Case 2: $i \notin T$. Then $T \notin W$, since i is part of all winning coalitions, there exists $R \subseteq N \setminus T$ such that $\pi(R) > \pi(T)$.

If $T \cap S^* \neq \emptyset$, then $T \subset S^*$. If $R \cap S^* \neq \emptyset$, then $R \subseteq S^* \setminus T$, then $\pi(S^* \setminus T) \geq \pi(R) > \pi(T)$, T loses the tournament. If $R \cap S^* = \emptyset$, $R \subseteq N \setminus S^*$, $\bar{P}$ is the maximal partition of $N \setminus S^*$, there is a set $T_1 \in \bar{P}$ such that $R \subseteq T_1$, $\pi(T_1) \geq \pi(R) > \pi(T)$, T loses the tournament.

If $T \cap S^* = \emptyset$, S^* is winning coalition, $\pi(S^*) > \pi(T)$, T does not defeat partition $\bar{P} \cup \{S^*\}$.

Consequently, the partition $P = \bar{P} \cup \{S^*\}$ is stable, which means the core is not empty.

($\Rightarrow$) We will prove the contrapositive version: Given a set of feasible coalitions $\mathbb{F}$, if there are two coalitions $A, B \in \mathbb{F}$ such that $A \cap B \neq \emptyset$ but $A \nsubseteq B$ and $B \nsubseteq A$, then there exist preferences defined on $\mathbb{F}$ and a power function π, such that the core is an empty set.

Note that if $A \cap B \neq \emptyset$ but $A \nsubseteq B$ and $B \nsubseteq A$, then $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Moreover, by definition 2, there is a partition $P_{B \setminus A} \subset \mathbb{F}$ of $B \setminus A$.

Let $K \in P_{B \setminus A}$ be a feasible coalition and note that $K \subset B$. Consider a power function π, such that:

$$\pi(A) > \pi(B) > \pi(K) > 0,$$

For any coalition S, $A \nsubseteq S$ and $K \nsubseteq S$, $\pi(S) = 0$.

Now, suppose that we have a set of preferences such that:

- For all $i \in A$: $A \succ_i T$ for all T, for all $i \in T$, $T \in \mathbb{F} \setminus \{A, B\}$.
- For all $i \in B$: $B \succ_i T$ for all T, for all $i \in T$, $T \in \mathbb{F} \setminus \{A, B, K\}$.
- For all $i \in A \cap B$: $B \succ_i A \succ_i T$, for all $i \in T$, $T \in \mathbb{F} \setminus \{A, B\}$.
- For all $i \in K$: $K \succ_i B \succ_i T$, for all $i \in T$, $T \in \mathbb{F} \setminus \{B, K\}$.

We show that the core is empty in this case.

For partition $P \in \Pi(N)$, if there is no winning coalition in P or $\pi(S) = 0$ for all $S \in P$, then coalition B defeats partition P.

For partition P, there is coalition $S \in P$, such that S is winning coalition, and $\pi(S) > 0$.
Case 1: $A \subset S$ and $B \nsubseteq S$. Any partition $P = (S, [N \setminus S])$ is defeated by coalition A, because $A \succ_i S$ for all $i \in A$. Thus, A wins the tournament in partition $(A, [S \setminus A], [N \setminus S])$.

Case 2: $B \subset S$ and $A \nsubseteq S$. Any partition $P = (S, [N \setminus S])$ is defeated by coalition B, because $B \succ_i S$ for all $i \in B$, and B wins the tournament in partition $(B, [S \setminus B], [N \setminus S])$.

Case 3: $A \cup B \subseteq S$. Any partition $P = (S, [N \setminus S])$ is defeated by coalition B, because $B \succ_i S$ for all $i \in B$. Thus, B wins the tournament in partition $(B, [S \setminus B], [N \setminus S])$.

Case 4: $S = A$. Any partition $P = (A, [N \setminus A])$ is defeated by coalition B, because $B \succ_i A$ for all $i \in A \cap B$. Thus, B wins the tournament in partition $(B, [A \setminus B], [N \setminus (A \cup B)])$.

Case 5: $S = B$. Any partition $P = (B, [N \setminus B])$ is defeated by coalition K, because $K \succ_i B$ for all $i \in K \subset B$. Thus, K wins the tournament in partition $(K, [B \setminus K], [N \setminus B])$. Therefore, P is not stable.

Case 6: $A \nsubseteq S$ and $B \nsubseteq S$, then $K \subset S$, $S \neq K$ because $K \notin W$ due to $A \subset N \setminus K$ and $\pi(A) > \pi(K)$. Any partition $P = (S, [N \setminus S])$ is defeated by coalition B, because $B \succ_i S$ for all $i \in B$, and B wins the tournament in partition $(B, [S \setminus B], [N \setminus (S \cup B)])$.

This shows that there are no stable partitions, and so the core is empty. □

Proof of Corollary 2

Proof. From Proposition 2, if the core exists in the tournaments, for any coalitions $A, B \in \mathbb{F}$ and $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$.

We prove by induction that $\mathbb{F} \subseteq C(H)$, such that network H is a line, and any agent i could be a node at the end of the line H.

For $n = 3$, If $\{1, 2, 3\}$ is feasible coalition, it is the same with the case that it is not a feasible coalition in the proof. Without loss of generality, assume $\{1, 2, 3\}$ is not feasible.

If the feasible coalitions include agent 1 is $\{1\}$, $\{1, 2\}$. Then the only feasible coalition include agent 3 is $\{3\}$, $\mathbb{F}$ is subset of connected coalitions of a line H with agent $1, 2, 3$ from left to right. A line with agent $2, 1, 3$ from left to right also works.

If the only feasible coalition include agent 1 is $\{1\}$, consider a line H with agent $1, 2, 3$ from left to right, $\mathbb{F} \subset C(H)$. A line with agent $1, 3, 2$ from left to right also works. From above, in any case, feasible coalitions $\mathbb{F}$ is a subset of the connected coalitions with agent $1, 2, 3$ arranged on a line, and any agent at the end of the line.

Suppose the result is true for n agents, then we prove for $n + 1$ agents. Pick an arbitrary agent a, assume the feasible include agent a is $S_1, S_2, \ldots, S_m$, $a \in S_i \cap S_j$ for any i, j, then $S_i \subset S_j$ or $S_j \subset S_i$. Without loss of generality, assume $S_i \subset S_2 \ldots, \subset S_m$.

If $|S_1| \geq 2$, there exists agent $b \in S_1$, such that in any feasible coalition $S \in \mathbb{F}$, $a \in S$ is equivalent with $b \in S$, then agent a, b could be seen as a union of agents in the tournaments and one node in network. From induction, H could be a line with agent a, b on the left.

If $|S_1| = 1$, there exists agent $b \in S_2$. For agents $\{2, \ldots, n, n+1\}$, from induction, there exists a line with agent b at the end, then construct line H with agent a connected to agent b at the end. The feasible coalitions $\mathbb{F}$ is subset of connected coalitions of line H.

From above, any agent a could be a node at the end of the line H. The results for $n+1$ is proved. □

Proof of Corollary 3

Proof. Suppose that the core is not empty for all preferences defined on $\mathbb{F}$ and all power functions. Let $T = \{S_1, S_2, \ldots, S_k\}$ is any family composed of potentially winning coalitions in $\mathbb{F}$, then $S_i \cap S_j \neq \emptyset$ for all $i, j \in [k]$.

By Proposition 2, the family T is totally ordered by the $\subset$ relation. Without loss of generality, we assume $S_1 \subset S_2 \subset \cdots \subset S_k$. Thus, $S_1 = \bigcap_{i \in [k]} S_i \neq \emptyset$. Therefore, $\mathbb{F}$ is a Helly family and an NTE exists, by Theorem 1. □

Proof of Proposition 3

Proof. Let S a family composed of potentially winning coalitions in $\mathbb{F}$. We proof that $\bigcap S \neq \emptyset$.
Case 1: There exists $R \in S$ such that $|R| = 1$. If $\{i\} \in S$, then $i \in R$ for all $R \in S$ by definition of joint business venture composed of potentially winning coalitions. Thus, $\{i\} \in \bigcap S$.
Case 2: $|R| = 2$ for all $R \in S$. Let A and B different joint business ventures in S. Then $|A \cap B| = 1$. Without loss generality suppose that $A \cap B = \{3\} \subseteq C$, thus $A = \{1, 3\}$ and $B = \{2, 3\}$ for some $1, 2 \in L$.

We show that $3 \in \bigcap S$. Suppose that there is coalition $D \in S$ such that $3 \notin D$. But $A \cap D \neq \emptyset$ and $D \cap B \neq \emptyset$, then $\{1, 2\} \subset D$. This contradicts that each coalition only has at most one labor business and one capital business in the power matching problem.

Therefore, by Theorem 1 the power matching problem has an NTE for all preferences and all power functions. □

Proof of Corollary 4

Proof. From Proposition 2, the core exists in the tournaments if and only if for any coalitions $A, B \in \mathbb{F}$ and $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$, which means the feasible coalitions $\mathbb{F}$ have a level structure.
($\Rightarrow$) We will prove the farsighted core exists only if the feasible coalitions have a level structure with less or equal than 2 levels by contradiction.

If feasible coalitions $\mathbb{F}$ have a level structure with 3 or more levels. There exists agent i, such that $i \in S_1 \subset S_2 \subset S_3$, S_1, S_2, S_3 are feasible coalitions.

Suppose we have a set of preferences such that: For all $a \in S_1$, $S_2 \succ_a S_3 \succ_a S_1$. For all $b \in S_2 \setminus S_1$, $S_3 \succ_b S_2$.

Assume the winning coalition are S_1, S_2, S_3.

For partition $P_1 = (S_3, [N \setminus S_3])$, there is chain of coalitions S_1, S_2 that defeats P_1.

For partition $P_2 = (S_2, [N \setminus S_3], [S_3 \setminus S_2])$, there is chain of coalitions $S_2 \setminus S_1, S_3$ that defeats P_2.

Thus, there is no farsighted stable partition.

($\Leftarrow$) We will prove the contrapositive version: Given a set of feasible coalitions $\mathbb{F}$, $\mathbb{F}$ have level structure with less or equal than 2 levels, then there exists farsighted core.

If the feasible coalitions have 1 level, the problem is trivial, any agent would be in a unique feasible coalition, there is no feasible deviation for agents. Thus the partition is farsighted stable.

If the feasible coalitions have 2 levels, agents belong to at most two feasible coalitions. Let $W = \{S_1, \ldots, S_m\}$ be the set of winning coalitions. By Definition 5, $S_i \cap S_j \neq \emptyset$ for all $i, j = 1, \ldots, m$. If $|W| \geq 3$, suppose S_1, S_2, S_3 are winning coalitions, without loss of generality, assume $S_1 \subset S_2 \subset S_3$, agent $a \in S_1$ belong to at least 3 feasible coalitions S_1, S_2, S_3, it is a contradiction. Thus, $|W| \leq 2$.

If $|W| = 1$, assume $W = S_1$. For any agent $a \in S_1$, if there exists no coalition $S_2 \in \mathbb{F}$, such that $a \in S_2$. Agents in S_1 have no other feasible coalition, partition $P = (S_1, [N \setminus S_1])$ is farsighted stable. If there exists $S_2 \in \mathbb{F}$, such that $a \in S_2$. For monotonicity of power, there is $S_2 \subset S_1$. Assume $\bar{P} \in \Pi(N \setminus S_1)$ is the maximal partition of $N \setminus S_1$, and partition $P = \bar{P} \cup \{S_1\}$. S_1 is winning coalition, if there exists a chain of coalitions $S^1, \ldots, S^k$ to defeat partition, there exists a subset of S_1 to deviate and become better off, assume it is S_2. Assume P^i is the partition after S^i deviates, then for agent $a \in S_2$, $S_2 \succ_a S_1$. S_2 is not winning coalition, there exists coalition $T \in \bar{P}$, such that $\pi(T) > \pi(S_2)$. If $S_2 \in P^k$, and $G_a(P^k) \succ_a G_a(P^{j-1})$, with $S_2 = S^j$. Then a subset T_1 of coalition T must deviate in the chain of coalitions, otherwise S_2 is not winning in P^k. But agents in T_1 have no incentive to deviate since it is not winning in P^k, it is a contradiction. Thus, partition $P = \bar{P} \cup \{S_1\}$ is farsighted stable.

If $|W| = 2$, assume $W = S_1$, and $S_1 \subset S_2$. The feasible coalitions for agents in S_1 are S_1, S_2. If $S_1 \succ_a S_2$, for any $a \in S_1$. Then partition $P = ([N \setminus S_1], S_1)$ is farsighted stable, for agents in S_1 have no incentive to deviate. If there exists $b \in S_1$, $S_2 \succ_b S_1$. Then partition $P = ([N \setminus S_2], S_2)$ is farsighted stable, for not all agents in S_1 have no incentive to deviate.

□

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