Generalized Consistent Ranking and the Formation of Self-enforcing Coalitions*

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Abstract

Agents endowed with powers compete for a divisible prize by forming coalitions. When a coalition wins, all non-members are eliminated. The winning coalition then divides the prize among its members according to a given sharing rule. We investigate the case where the sharing rule satisfies a property we call *consistent ranking*. Sharing rules that satisfy consistent ranking ensures that agents' ranking of competing coalitions coincide. Sharing rules such as equal and proportional sharing satisfy this property. We also examine a larger class of sharing rules that satisfy a property we call *generalized consistent ranking* where agents are able to rank coalitions even though the sharing rule does not satisfy consistent ranking. For instance, a convex combination of equal and proportional sharing, which we call *combination sharing*, violates consistent ranking but satisfies generalized consistent ranking under certain conditions.

For these different sharing rules, we characterize rules on choosing coalitions (called *transition correspondence*) that satisfy two main axioms: *self-enforcement*, which requires that no further deviation happens after a coalition has formed, and *rationality*, which requires that agents pick the coalition that gives them their highest payoff.

We find that a transition correspondence that satisfies self-enforcement and rationality always exists for sharing rules that satisfy generalized consistent ranking (and hence, consistent ranking). In order to find the maximal domain of games for combination sharing to satisfy generalized consistent ranking, we need to restrict the configuration of powers in the society or restrict the convex combination parameter between equal sharing and proportional sharing.

Keywords: Coalition Formation, Sharing Rules, Generalized Consistent Ranking, Self-enforcement.

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1 Introduction

Forming groups confers to its members certain advantages that cannot be appropriated had its members acted individually. However, conflicts may arise within the group when there are disagreements in dividing the prize or effort among its members. Thus, specifying the appropriate sharing rule to divide an economic surplus (or cost) provides the incentive structure that encourages cooperation or discourages conflict within coalitions (Moulin [31]).[1] These incentives to maintain stability in coalitions, however, are altered when some agents in the society are more influential or powerful than others (Jandoc and Juarez [25, 26]; Piccione and Razin [33]; Jordan [28]).

While there has been a large and growing literature on coalition formation with externalities focusing on static equilibria (e.g., Bogomolnaia and Jackson [8], Papai [32], Ehlers [17], Bloch and Dutta [7], Chatterjee et al. [13], Pycia [34], Romero-Medina [10], Banerjee et al. [4]) and dynamic equilibria (e.g., Greenberg [22], Chwe [14], Bloch [6], Xue [42], Arnold and Schwalbe [3], Diamantoudi and Xue [15], Ray and Vohra [38], Iñarra et al. [24]), there are far fewer studies that examine how the interconnectedness of sharing-rules (payoffs) and the power of agents (ability to form winning coalitions) affect the stability of coalitions in dynamic settings. An important advance in the literature to address this gap stems from Acemoglu, Egorov and Sonin [1, 2] (henceforth, AES). AES provides a coalition formation model where agents endowed with different powers compete for a divisible prize by forming coalitions with other agents for infinitely many rounds. In each round, a *winning coalition* will form where agents inside this coalition have combined powers that are higher than the combined powers of the rest of the society. Agents who are not part of this winning coalition are "killed" in the sense that they cannot participate in succeeding rounds of the game.[2]. If there is a winning coalition that forms without subsequent deviations into subcoalitions after several rounds, this winning coalition is deemed as the *ultimate ruling coalition*.[3] Members of the ultimate ruling coalition will divide this prize among themselves according to a fixed and predetermined sharing rule.

AES focused on the case when agents in the ultimate ruling coalition divide the prize in proportion to their power inside this coalition. We call this type of sharing

[1] The applications are wide-ranging, for instance, appropriating gains and costs from managing global public goods (Carraro [9]), cost and revenue sharing in irrigation networks (Jandoc, Juarez and Roumasset [27]), and market design (Roth [39]), to name a few.

[2] A real world example of this setting is the purge in the Russian Politburo documented in Acemoglu, Egorov and Sonin [1].

[3] Elsewhere in this paper, we interchange the term ultimate ruling coalition with the term *limit coalition* or *final coalition*.

rule *proportional sharing*. AES characterizes which coalitions will form at the very first round of the game that will never disintegrate into subcoalitions in subsequent rounds. However, this characterization by AES does not extend to different types of sharing rules, such as when the prize is divided equally among coalition members (we call this type of sharing rule *equal sharing*).[4] Jandoc and Juarez [25] extends the characterization to equal sharing and to the case when power accumulates over time.

These two sharing rules, equal and proportional sharing, share the property that they do not create disagreements among agents when faced between a choice of joining competing coalitions. We call this property *consistent ranking* because these sharing rules induce agents to have the same ordinal ranking over coalitions in which they could possibly join. Equal sharing satisfies consistent ranking because agents in the intersection of competing coalitions will unanimously prefer to join the coalition with the smaller size. Proportional sharing satisfies consistent ranking because agents in the intersection of competing coalitions will unanimously prefer to join the coalition with the least total power. In both cases, agents increase their share of the prize by joining these coalitions.

However, there are also sharing rules that create disagreements among agents on their preferred coalitions. For instance, a convex combination of equal and proportional sharing (call this *combination sharing*) may induce some agents in the intersection of competing coalitions to prefer the smaller-sized coalition, while others may prefer to be in the lower-powered coalition. This situation then creates tension between coalition size and power. This sharing rule has its roots in the *Sen share* (Sen [40]) popular in the literature of team production and moral hazard.[5] Combination sharing does not satisfy the consistent ranking property.

There is, nevertheless, a larger class of sharing rules that still enables the agents' ranking to align on certain coalitions. We call this property *generalized consistent ranking* and this property will be crucial for agents to agree on competing coalitions. Generalized consistent ranking is related to the notion of *pairwise alignment in preference profiles* (Pycia [34]) which says that agents' preferences are pairwise aligned if any two agents rank coalitions that contain both of them in the same way. Pycia [34] shows that pairwise alignment is a necessary and sufficient condition to establish the existence of a stable coalition structure for all preference profiles. It is also related to

[4]Equal sharing is widely used not only on theoretical grounds but also for practical purposes such as inheritance bequest (Erixson and Ohlsson [18]).

[5]The Sen share is a convex combination of an equal share of the surplus among productive agents and a proportional share according to an agent's labor to the total labor supplied in the economy. Fabella [19,20], shows that under increasing returns to scale the Sen share of the production surplus can support Pareto optimal production.

the condition of *common ranking property* by Farrell and Scotchmer [21] that guarantees stability of the core in hedonic games. Also related to consistent ranking is the *top coalition property* of Banerjee, et al. [4], that requires for any nonempty subset of agents S, there exists a coalition $T \subseteq S$ such that all members of coalition T prefer it to any other coalition that contains some (or all) members of S.

The main objective of this paper is to examine how different sharing rules affect the manner in which coalitions form when agents have heterogeneous power. "Power" can emanate from various sources–for instance wealth, military might or political influence–and is exogenously given in the model.[6]

Similar to AES, we employ an axiomatic approach to find rules to choose coalitions (which we call *transition correspondences*) that satisfy two main axioms: *self-enforcement* and *rationality*. In our model, we allow for the possibility of factions within the winning coalition to secede from this coalition and form their own winning coalition in the next round. Since agents are forward-looking, they prefer to be part of the ultimate ruling coalition to get a share of the prize. Our axiom of self-enforcement requires that no subcoalition of the winning coalition be powerful enough to encourage further deviations. Self-enforcement is a robustness property that ensures that the winning coalition that forms never disintegrates afterwards.

The notion of self-enforcement is related to several stability concepts in the literature. First, it is related to the consistency requirement that has been the critique of traditional concepts such as the core.[7] For instance, the *modified core* concept of Ray [35], the *core in "stable standards of behavior"* of Greenberg [22] and the *coalition proof Nash equilibrium* of Bernheim, Peleg and Whinston [5] require that objections to the grand coalition from deviating subcoalitions must demand that these subcoalitions are themselves stable. Second, there is also the related issue of *myopia*, where agents object to a deviation because it hurts them in the short term, without realizing that there may be further gains once final outcomes are realized in the future. This issue inspired the development of notions such as *farsighted blocking* that takes into account the sequence of deviations that will lead to an improvement for the coalitions in some "final state". This was formalized in the notion of *farsighted stability* attributed to Chwe [14].[8]. Stability under this concept requires that the "final coalition" makes all

[6]Since agents are given exogenous power, we avoid the "joint-bargaining paradox" where, in a pure bargaining situation, an individual can be made worse-off by negotiating as a member of a group than by negotiating alone (Harsanyi [23]; Chae and Heidhues [11]; Chae and Moulin [12]).

[7]The consistency critique of the traditional concept of the core stems from the requirement that the subcoalitions that object to the formation of the grand coalition must themselves be tested against further objections (see Ray and Vohra [36] for a discussion of this consistency critique).

[8]This is also introduced by Ray and Vohra [36] in the "sequential blocking" approach in equilibrium

the agents not worse off than in an initial forming coalition (e.g. grand coalition). Self-enforcement requires the same criteria that all agents will be better off in the final coalition as opposed to any other past or future coalitions that may form. In our context, although agents potentially increase their expected payoffs from deviating into an intermediate subcoalition, the possibility that they will eventually be excluded from the final coalition (and get no share of the prize) will discourage them from certain types of deviations.[9] In addition, rationality requires that agents choose to join the coalition that gives them their highest payoff among self-enforcing coalitions. Rationality is related to the traditional axiom of coalitional stability, where no coalition will have the incentive to deviate.[10]

In the next subsection, we provide an illustrative example of how our coalition formation game works and how sharing of the prize shapes the coalitions that form.

1.1 An illustrative example

Suppose our society is composed of 6 agents $\{1, 2, 3, 4, 5, 6\}$ with power profile $\pi = [29, 26.5, 21, 2.49, 2.4, 2.39]$. The game is played in discrete rounds and for each round the prize to be divided by the winning coalition is $I = 1$. A coalition's power is simply the sum of the agents' powers inside it. A winning coalition is a coalition such that it has more than 50% of the total power in the society at any given round. Since agents are farsighted, they only care about their payoffs by being a member of the ultimate ruling coalition. If an agent is part of a coalition S, his share of the prize under equal sharing will be $\frac{1}{|S|}$, under proportional sharing it is his power divided by the power of coalition S (that is, $\frac{\pi_i}{\pi(S)}$), under combination sharing it is $\lambda\frac{1}{|S|} + (1-\lambda)\frac{\pi_i}{\pi(S)}$ for some $\lambda \in (0, 1)$.

Case 1: Sharing rule satisfies consistent ranking.

First, no coalition of size 1 is winning and therefore is incapable of forming. Although there are winning coalitions of size 2, we argue that this will not satisfy self-enforcment.[11] To see this, suppose coalition $\{1, 2\}$ forms. This coalition is winning since their combined power (55.5) is higher than the rest of society composed of agents $3, 4, 5$ and 6 (with combined power of 28.28). After coalition $\{1, 2\}$ forms, the non-

binding agreements. An excellent review of these issues is found in Ray and Vohra [37].

[9] Using a laboratory experiment, Jandoc and Juarez [26] show that agents do not display farsighted behavior when playing a simplified version of the game informed by the model of this paper.

[10] This is related to immunity to group manipulations in models discussed by Bogomolnaia and Jackson [8], Ehlers [17], Juarez [30], Papai [32].

[11] The precise definition of self-enforcement is stated in Axiom 1.

winning agents $3, 4, 5$ and 6 are killed and agents 1 and 2 continue on the next round. Left alone to themselves, agent 1 can now kill agent 2 since he has the higher power. Since agent 2 is farsighted, he will never agree to form $\{1, 2\}$. This will be true for any coalition of size 2.

Furthermore, note that there are several coalitions of size 3 that are winning but only $\{1, 2, 3\}$ is self-enforcing. This is because if $\{1, 2, 3\}$ forms and agents 4,5, and 6 are killed, then there can be no dictator among them and a deviation to a 2-person coalition is not feasible following the argument outlined above. On the other hand, take the case of winning coalition $\{2, 3, 5\}$. This will not be self-enforcing since when agents 1,4 and 6 are killed, agent 2 can be a dictator and can deviate from $\{2, 3, 5\}$.

Following the same arguments, it is straightforward to show that the coalitions $\{1, 2, 4, 5\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 4, 6\}$ and $\{2, 3, 4, 5, 6\}$ are self-enforcing since they don't contain subcoalitions that are self-enforcing. In the same manner, the grand coalition $\{1, 2, 3, 4, 5, 6\}$ is not self-enforcing since it can deviate to either one of the self-enforcing coalitions mentioned.

With proportional sharing, the coalition $\{2, 3, 4, 5, 6\}$ will be preferred by everyone inside this coalition over any other possible self-enforcing coalitions. For instance, given the choice between $\{2, 3, 4, 5, 6\}$ and $\{1, 2, 3\}$ agents 2 and 3 (the agents common in these coalitions) will prefer the former since their share of the prize ($\frac{26.5}{54.78}$ and $\frac{21}{54.78}$ for agents 2 and 3, respectively) is higher than the latter coalition ($\frac{26.5}{76.5}$ and $\frac{21}{76.5}$ for agents 2 and 3, respectively). Hence, under proportional sharing, rationality implies that the coalition $\{2, 3, 4, 5, 6\}$ should form.

On the other hand, under equal sharing the coalition $\{1, 2, 3\}$ should form because the agents in this coalition will get a higher share $\frac{1}{3}$ than in any other possible self-enforcing coalition.

Case 2: Combination sharing.

Recall that combination sharing is a convex combination of equal sharing and proportional sharing, $\lambda\frac{1}{|S|} + (1 - \lambda)\frac{\pi_i}{\pi(S)}$ for some combination sharing parameter $\lambda \in (0, 1)$, We will show in Example 1 later that when the combination sharing parameter $\lambda = 0.54$, there will be disagreement among agents on the preferred coalition. For instance, comparing coalitions $\{1, 2, 3\}$ and $\{2, 3, 4, 5, 6\}$, the share of agent 2 is higher in $\{2, 3, 4, 5, 6\}$ and the share of agent 3 is higher in $\{1, 2, 3\}$. Unlike in the previous case of equal or proportional sharing where these agents agree on the preferred coalition, in the case of combination sharing there may be no such agreement.

1.2 Overview of the results

Section 4 of the paper presents the main results. Proposition 1 characterizes the unique transition correspondence that is rational and self-enforcing under sharing rules that satisfy consistent ranking.[12] Our proof is based on the key Lemma 1, which highlights the importance of transition correspondences that do not necessarily satisfy rationality but meet a weaker requirement, *minimalistic*, introduced in Section 3. Under consistent ranking, the transition correspondence chooses the highest ranked coalition among self-enforcing and winning coalitions. In particular, for equal sharing, the smallest-sized self-enforcing coalition will be chosen, whereas for proportional sharing, the least-powered self-enforcing coalition is chosen.

Proposition 2 gives the necessary and sufficient conditions for the existence of a transition correspondence that satisfies self-enforcement and rationality to a larger class of sharing rules that satisfy a property which we call "generalized consistent ranking".

Under combination sharing, Propositions 3 and 4 characterize the unique transition correspondences that are self-enforcing and rational for restricted classes of coalition formation games. Indeed, we show that the class of games with power vectors that meet size-power monotonicity (where larger coalitions have higher power) is the maximalist domain that works for any combination sharing rule (Proposition 3). Alternatively, for a fixed combination sharing rule, Proposition 4 characterizes the maximalist domain of games where combination sharing will satisfy generalized consistent ranking, guaranteeing the existence of a unique transition correspondence that satisfies self-enforcing and rationality, which we also characterize. Such a domain is substantially larger than the domain found in Proposition 3. Furthermore, such a correspondence may select the smallest-sized self-enforcing coalition for some games, the least-powered self-enforcing coalition for other games, and (perhaps surprisingly) just the right "compromise" coalition for even other games.

2 The model

Consider the set $N = \{1, \ldots, n\}$ of initial agents who are endowed with powers $\pi = [\pi_1, \ldots, \pi_n]$, respectively.[13] A coalition S is a subset of N, that is, $S \subseteq N$. The set of coalitions are all possible subsets of N, denoted by 2^N. A coalition formation game

[12] A transition correspondence is a mapping that defines which coalitions form over time. The precise definition is given in Section 2.

[13] For convenience, the power vector $\pi \in \mathbb{R}^N_+$ can be normalized such that $\sum_{i \in N} \pi_i = 1$.

is a pair (S, π) where $S \subseteq N$ and $\pi \in \mathbb{R}^S_+$. For convenience, we restrict the game (N, π) such that π has no ties in the power of any pair of coalitions, that is, $\pi(V) \neq \pi(T)$ for any $V, T \subset N$.[14] The set of coalition formation games is denoted by $\mathbf{G}$. We assume that power is additive, that is, the power of coalition S is the sum of all powers of the agents inside the coalition, $\pi(S) = \sum_{i \in S} \pi_i$.[15] We denote by π_S the restriction of the vector $\pi \in \mathbb{R}^N_+$ over coalition S.

Definition 1 *Given a game (T, π), the* ***set of winning coalitions*** *is*

$$W_{(T,\pi)} = \{S \subseteq T | \pi(S) > \alpha\pi(T)\}.$$

Without loss of generality, we assume that $\alpha = 0.5$, requiring winning coalitions to have relative power larger than 50%. This is equivalent to the requirement that a winning coalition S to have power $\pi(S) > \pi(T \setminus S)$ for $S \subseteq T$.[16]

There is a prize $I \in \mathbb{R}_+$ that will be divided by the agents according to a sharing rule that is fixed throughout the game. Without loss of generality, we assume $I = 1$.

Definition 2 *A* ***sharing rule*** *is a function $\xi : \mathbf{G} \to \mathbb{R}^N_+$ such that:*

i. If $k \notin S$, then $\xi_k(S, \pi) = 0$.

ii. $\sum_{i \in S} \xi_i(S, \pi) = 1$, and

iii. (Cross-Monotonicity) If $(S, \pi) \in \mathbf{G}$, $i \in T \subset S$, and $\pi_i > 0$, then $\xi_i(T, \pi_T) > \xi_i(S, \pi)$.

Agents who are not part of the forming coalition gets no share of the prize (condition *i*) and the share of the agents sum to 1 (condition *ii*). Cross-monotonicity of the sharing rule requires that the share of the prize of agents in the subcoalition $T \subset S$ will be higher in T than in the larger coalition S (condition *iii*)[17].

Throughout the paper we devote special attention to commonly used sharing rules such as equal sharing, proportional sharing, or a convex combination of the two.[18]

[14] However, this is a weak condition because games that do not satisfy this property has a Lebesgue measure equal to zero (See Jandoc and Juarez [25], Acemoglu et al. [1]).

[15] Juarez and Vargas [29] considers a more general version where power can be any arbitrary monotonic function.

[16] Our results can be easily adapted to require winning coalitions to have relative power larger than 50%, that is, $\alpha \in [0.5, 1)$.

[17] Cross-monotonicity is related to the *axiom of smaller coalitions* in Shenoy [41]. Dimitrov and Haake [16] investigated coalition formation for games with this property.

[18] Note that these three sharing rules are cross-monotonic.

That is, if $i \in S$, then the share of agent i when S is winning and the power profile is π equals:

$$\xi_i(S,\pi) = \begin{cases} \dfrac{1}{|S|} & \text{if equal sharing (ES)} \\ \dfrac{\pi_i}{\pi(S)} & \text{if proportional sharing (PR)} \\ \lambda \cdot \dfrac{1}{|S|} + (1-\lambda) \cdot \dfrac{\pi_i}{\pi(S)},\ \lambda \in (0,1) & \text{if combination sharing (CS}^{\lambda}\text{)} \end{cases}$$

Suppose agents i and j belong to the intersection of coalitions S and T. A sharing rule satisfies consistent ranking if whenever agent i prefers S over T, then agent j also prefers S over T. In other words, between competing coalitions, a coalition S is picked if all agents in the intersection unanimously pick S over a competing coalition.

Definition 3 (Consistent Ranking (CR)) *The sharing rule ξ satisfies **consistent ranking (CR)** if for any two agents i and j, and coalitions S and T such that $i, j \in S \cap T$, if $\xi_i(S,\pi) > \xi_i(T,\pi)$, then $\xi_j(S,\pi) > \xi_j(T,\pi)$.*

If the sharing rule ξ satisfies consistent ranking, then there exists a ranking $R^{\xi} : \mathbf{G} \to \mathbb{R}$ for the society that coincides with individual rankings on coalitions to which the individual belongs. That is, for any coalitions S and T such that $S \cap T \neq \emptyset$, we have that $R^{\xi}(S,\pi) > R^{\xi}(T,\pi) \Leftrightarrow \xi_i(S,\pi) > \xi_i(T,\pi)$ for any $i \in S \cap T$.

Equal sharing and proportional sharing satisfy consistent ranking. Under equal sharing, agents' share increases as they move to coalitions of smaller sizes; therefore, $R^{ES}(S,\pi) = \frac{1}{|S|}$ is an example of a consistent ranking for equal sharing. Similarly, under proportional sharing, agents' share increases as they move to coalitions of lower power; therefore, $R^{PR}(S,\pi) = \frac{1}{\pi(S)}$ is an example of a consistent ranking for proportional sharing.

We define a *transition correspondence* that maps from the set of coalition formation games to the set of all winning coalitions.

Definition 4 *A* **transition correspondence** *is a continuous correspondence $\phi : \mathbf{G} \to 2^N$ such that $\forall\ (X,\pi) \in \mathbf{G}$: $\phi(X,\pi) \subseteq W_{(X,\pi)}$.*[19]

The transition correspondence ϕ selects all winning coalitions emanating from the game (S,π). For discrete rounds t, where $t = 0, 1, \dots$, the evolution of the coalition formation game at every round depends on the transition correspondence, starting with

[19] A correspondence is continuous if for any sequence of power vectors $\pi^1, \pi^2, \dots \to \pi^*$ where $S \in \phi(N,\pi^i)\ \forall i$ and S is winning in π^*, then $S \in \phi(N,\pi^*)$.

the game (S^0, π^0) at time 0 and throughout succeeding rounds $(S^1, \pi^1), (S^2, \pi^2), \ldots,$ where $S^t \in \phi(S^{t-1}, \pi^{t-1})$ is a chosen coalition at time t and $\pi^t = \pi^{t-1}_{S^t}$ is their respective power. Furthermore, we assume that agents are killed, $S^t \subseteq S^{t-1}$ for any $t \geq 1$. That is, only agents who were part of the winning coalition at time $t-1$ could participate at time t.[20]

Since coalition sizes do not increase over time, the sequence $(S^0, \pi^0), (S^1, \pi^1), \ldots$ converges in at most n steps. We call this limit (S^∞, π^∞). The prize is only given to the limit coalition and agent $i \in S^\infty$ get his share of the prize $\xi_i(S^\infty, \pi^\infty)$. Again, agents in the coalition formation game are infinitely forward-looking and they only care about his share of the prize in the limit.

3 Axioms

The first main axiom, self-enforcement, ensures that a transition correspondence maps to coalitions that do not have the incentive nor the power to deviate in future rounds of the game.

Axiom 1 (Self-enforcement (SE)) *The transition correspondence ϕ is **self-enforcing (SE)** if for any game $(X, \pi) \in \mathbf{G}$ and $S \in \phi(X, \pi)$, then $S \in \phi(S, \pi_S)$.*

When there is no confusion, given a transition correspondence ϕ and a game (S, π), we say that the coalition S is self-enforcing if $S \in \phi(S, \pi_S)$.

Self-enforcement requires that if coalition S is part of the set chosen by the transition correspondence $\phi(X, \pi)$ given a starting game (X, π), then it is also part of the set chosen again by the same transition correspondence in a game where only coalition S survives.

Since the sharing rule is cross-monotonic, we expect that in the presence of self-enforcing and winning coalitions that are strict subsets of the grand coalition, the grand coalition will not be chosen, since all of the agents gain by choosing its subset. This is reflected in the definition of a minimalistic transition correspondence.

Axiom 2 (Minimalistic (MIN)) *The transition correspondence ϕ is **Minimalistic (MIN)** if for a game $(S, \pi) \in \mathbf{G}$ such that there exists $T \subset S$, where $T \in \phi(T, \pi_T)$ and $T \in W_{(S,\pi)}$, then $S \notin \phi(S, \pi)$.*

[20] We impose no restriction in which coalition from $\phi(S^{t-1}, \pi^{t-1})$ will be selected. This allows our results to be more robust, since the evolution of the game includes any potential path of coalitions such that $S^t \in \phi(S^{t-1}, \pi^{t-1})$ for all t.

There exists a large class of transition correspondences that satisfy SE and MIN, including the transition correspondence that chooses all subsets of the grand coalition that are winning and self-enforcing. The following definition would allow us to narrow down this large class by comparing different transition correspondences based on the coalitions that they choose.

Definition 5 (Superiority) *Consider two transition correspondences ϕ and $\hat{\phi}$. We say that ϕ is **superior** to $\hat{\phi}$ if for any game (X, π), $T \in \hat{\phi}(X, \pi)$ and $S \in \phi(X, \pi)$ it follows that $\xi_i(S, \pi_S) \geq \xi_i(T, \pi_T)$ for all $i \in T \cap S$.*

If a transition correspondence is superior to another, then it always picks outcomes that common agents in intersecting coalitions prefer compared to the coalitions picked by other competing transition correspondences.

Axiom 3 (Rationality (RAT)) *The transition correspondence ϕ is **rational (RAT)** if for any $S \in 2^N$, for any $T \in \phi(S, \pi)$ and for any $Z \subset S$, $T \neq Z$, such that $Z \in W_{(S,\pi)}$ and $Z \in \phi(Z, \pi_Z)$, we have that $Z \notin \phi(S, \pi) \Leftrightarrow \xi_i(T, \pi_T) > \xi_i(Z, \pi_Z)$ $\forall i \in T \cap Z$.*

Rationality implies that agents prefer to join self-enforcing coalitions that give them a larger share of the resource. This is similar to other notions of coalitional stability previously discussed in the literature, where a coalition is chosen if it cannot be blocked by another coalition that is winning and self-enforcing. Note that the cross-monotonicity of the sharing rule implies that if a transition correspondence satisfies RAT, then it also satisfies MIN.

4 Results

4.1 Result with Consistent Ranking

Let the transition correspondence ϕ^* be defined over the set $\mathbf{G}$ as follows:

$$\phi^* (S, \pi) = \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M, \pi_M) \tag{1}$$

where $Q(S, \pi) = \{T \subset S \mid T \in W_{(S,\pi)}, T \in \phi^*(T, \pi_T)\}$

This transition correspondence defines for the game (S, π) a set $Q(S, \pi)$ of proper subcoalitions, which are both winning in S and self-enforcing. It picks the coalition that yields the highest rank for the agents in the intersection of all the coalitions contained in $Q(S, \pi)$. If $Q(S, \pi)$ is empty then it picks coalition S itself. Thus, $\phi^*(S, \pi_S) \neq \emptyset$ and ϕ^* is well defined.

Proposition 1 *Let ξ be a sharing rule that satisfies consistent ranking, and let ϕ be a transition correspondence that is self-enforcing on* $\mathbf{G}$. *Then the following are equivalent:*

i. *ϕ is superior to any other transition correspondence that is self-enforcing and minimalistic,*

ii. *ϕ is rational,*

iii. $\phi = \phi^*$.

We note that AES's main result has a similar characterization to parts *ii* and *iii* under proportional sharing. This proposition shows that AES's result actually extends to a larger class of sharing rules that satisfy consistent ranking. The proof of this result is provided in Appendix A. The proof relies on using a key observation that any two self-enforcing and minimalistic transition correspondences have the same sets of self-enforcing coalitions. This is stated in Lemma 1. This observation greatly simplifies the proof provided by AES even for proportional sharing.

Lemma 1 *Consider the self-enforcing and minimalistic transition correspondences ϕ and $\tilde{\phi}$ for the sharing rules ξ and $\tilde{\xi}$, respectively. Then, for a given power vector π, the sets of coalitions that are self-enforcing coincide. That is,*

$$\{S | S \in \phi(S, \pi)\} = \{T | T \in \tilde{\phi}(T, \pi)\}.$$

Lemma 1 is a consequence of the fact that the set of self-enforcing coalitions only depends on the power profile of the agents and the cross-monotonicity of the sharing rule.

4.2 Results with Generalized Consistent Ranking

In this section, we show that the assumption of consistent ranking in Proposition 1 can be extended to a larger class of sharing rules that satisfies a property which we call "generalized consistent ranking". Here we discuss the necessary and sufficient conditions for the existence of transition correspondence that satisfy our desirable properties.

By Lemma 1, it is clear that with the assumption of cross-monotonicity, the set of coalitions produced by any self-enforcing and minimalistic transition correspondence coincide. Hence, we can define for any game (N,π) the set $SEC(N,\pi)$ which contains self-enforcing and winning coalitions. With the set $SEC(N,\pi)$ and the fixed sharing rule ξ, we can also define another set $UD^{\xi}(N,\pi)$, or the set of undominated stable coalitions in (N,π). Definition 6 provides the precise definitions of these two sets.

Definition 6 *Consider the game $(N,\pi) \in \boldsymbol{G}$ and a transition correspondence ϕ:*

i. The set

$$SEC(N,\pi) = \{T \subseteq N \mid T \in \phi(T,\pi_T) \text{ and } T \in W_{(N,\pi)}\}$$

is the set of coalitions that are self-enforcing and are winning in the game (N,π).

ii. The set

$$\begin{aligned} UD^{\xi}(N,\pi) = \{T \subseteq N \mid T \in SEC(N,\pi),\ \nexists\ S \in SEC(N,\pi) \\ \text{such that } \xi_i(S,\pi_S) > \xi_i(T,\pi_T) \forall i \in S \cap T\} \end{aligned}$$

is the set of ***undominated stable coalitions*** *in (N,π).*

Example 1 *Consider the game*

$$(N,\pi) = (\{1,2,3,4,5,6\},[29,26.5,21,2.49,2.4,2.39])$$

as described in Section 1.1.

The self enforcing coalitions that are contained in (N,π) are the singletons $\{1\}$, $\{2\}$, $\{3\}$—which are non-winning in the game (N,π)—as well as the winning coalitions $\{1,2,3\}$, $\{1,2,5,6\}$, $\{1,2,4,5\}$, $\{1,2,4,6\}$, and $\{2,3,4,5,6\}$. To see why these non-singleton winning coalitions are self-enforcing, note that the power of agents 3, 4, 5 and 6 together is less than the power of agent 1. Therefore, any self-enforcing coalition that contains agent 1 should also contain agent 2, otherwise 1 can deviate to form a singleton. In order for 1 not to be a dictator in a game that contains agents 1 and 2, the game must contain agents whose combined powers should exceed 2.5. Therefore, any self-enforcing coalition that contains agents 1 and 2 should include either agent 3 alone or any two agents from agents 4, 5 and 6. On the other hand, note that if a self-enforcing coalition does not contain agent 1, then it must be either the coalition $\{2,3,4,5,6\}$ or a subset of it. However, only the coalition $\{2,3,4,5,6\}$ is self-enforcing because any subcoalition of size 2, 3 or 4 has a dictator. Therefore, $\{2,3,4,5,6\}$ is the only self-enforcing coalition that does not contain agent 1. Hence, the set $SEC(N,\pi)$

defined over this game is

$$SEC(N,\pi) = \big\{\{1,2,3\},\{1,2,5,6\},\{1,2,4,5\},\{1,2,4,6\},\{2,3,4,5,6\}\big\}.$$

Let the prize be divided by the proportional sharing rule. Notice that the coalition $\{2,3,4,5,6\}$ *is preferred by the intersecting agents in this coalition against any other coalition in the set* $SEC(N,\pi)$*. For instance, agents* 2 *and* 3 *prefer* $\{2,3,4,5,6\}$ *over the coalition* $\{1,2,3\}$*; Agents* 2*,* 5 *and* 6 *prefer* $\{2,3,4,5,6\}$ *over* $\{1,2,5,6\}$*, and so forth. On the other hand, if we compare agents in coalition* $\{1,2,3\}$*, for instance, only agent* 1 *wants to belong to* $\{1,2,3\}$ *but agents* 2 *and* 3 *want to form* $\{2,3,4,5,6\}$ *since they get a higher share of the prize in this coalition. The same is true for the other coalitions in* $SEC(N,\pi)$ *except for the coalition* $\{2,3,4,5,6\}$*. Hence the set* $UD^{\xi}(N,\pi)$ *defined over this game with proportional sharing is*

$$UD^{PR}(N,\pi) = \big\{\{2,3,4,5,6\}\big\}.$$

Now consider the combination sharing rule with $\lambda = .5$*. Note that common agents in* $\{1,2,3\}$*, when compared to coalition* $\{2,3,4,5,6\}$ *do not agree to stay in* $\{1,2,3\}$ *because while agent* 3 *prefers to stay in* $\{1,2,3\}$*, agent* 2 *wants to move to* $\{2,3,4,5,6\}$*. This is because*

$$\xi_2(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.34$$

$$\xi_3(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.29$$

and

$$\xi_2(\{1,2,3\},[29,26.5,21]) = 0.33$$

$$\xi_3(\{1,2,3\},[29,26.5,21]) = 0.30$$

The share of agent 2 is higher in the coalition $\{2,3,4,5,6\}$ *while the share of agent 3 is higher in the coalition* $\{1,2,3\}$*. However, intersecting agents in the coalition* $\{1,2,5,6\}$ *prefer this coalition to any other coalition in* $SEC(N,\pi)$ *with this combination sharing rule with* $\lambda = 0.5$ *and there is no other coalition in* $SEC(N,\pi)$ *with such a characteristic. Therefore, the set* $UD^{\xi}(N,\pi)$ *defined over this game with combination sharing with* $\lambda = 0.5$ *is*

$$UD^{CS^{0.5}}(N,\pi) = \big\{\{1,2,5,6\}\big\}.$$

However, there can be instances when the set $UD^{\xi}(N,\pi)$ *will be empty. This is true, for example, when we have combination sharing but* $\lambda = 0.54$*. Here, agents in each of the* *coalitions in $SEC(N,\pi)$ disagree on preference. For example, comparing $\{1,2,5,6\}$ and $\{1,2,3\}$, agent 1 prefers $\{1,2,5,6\}$ while agent 2 prefers $\{1,2,3\}$. This is true when comparing any pair of coalitions in $SEC(N,\pi)$. Therefore, in this game with combination sharing and $\lambda = 0.54$, $UD^{CS^{0.54}}(N,\pi) = \emptyset$.*

We now define generalized consistent ranking.

Definition 7 (Generalized consistent ranking (GC)) *A sharing rule ξ satisfies **generalized consistent ranking (GC)** at the game $(N,\pi) \in \mathbf{G}$ if the set $UD^{\xi}(N,\pi) \neq \emptyset$.*

Generalized consistent ranking requires that at the game (N,π), there exists an undominated stable game that generates coalitions where agents prefer the coalition in question over any other coalition in $SEC(N,\pi)$. As we have shown in Example 1, there are sharing rules that make the set $UD^{\xi}(N,\pi)$ empty. Note that if a sharing rule satisfies consistent ranking, then it satisfies GC. This is because the agents share the same ordinal ranking R^{ξ} over games. Therefore, $UD^{\xi}(N,\pi)$ coincides with the set of coalitions generated by games that maximize R^{ξ}.

The class of sharing rules that satisfy generalized consistent ranking contains a large class of important sharing rules not covered by consistent ranking. The definition of generalized consistent ranking only imposes restrictions only on coalitions belonging to $SEC(N,\pi)$. Thus, for instance, a sharing rule will meet GC if it satisfies consistent ranking within the class of games containing coalitions from $SEC(N,\pi)$ (and does not necessarily satisfy consistent ranking for games containing coalitions outside $SEC(N,\pi)$). One example of such sharing rule might split the resource in proportion (or equally) within the class of games with coalitions from $SEC(N,\pi)$, and use combination sharing on games containing coalitions outside of $SEC(N,\pi)$.[21]

Alternatively, note that the sharing rule does not need to satisfy consistent ranking within the class of games containing coalitions contained in $SEC(N,\pi)$. For instance, consider a convex combination of dictatorial sharing and proportional, where 90% of the resource is allocated to a single agent following a priority ordering and the remaining 10% of the resource is allocated to all the agents in proportion to their power. For instance, if the priority ordering is $1 \succ 2 \succ \cdots \succ n$, then for all the games in the set $SEC(N,\pi)$ that contain agent 1, 90% of the resource is given to agent 1 and the remaining 10% is split between all the agents in proportion to their power. For all the games in the set $SEC(N,\pi)$ that contain agent 2 but do not contain agent 1, 90% of the resource is given to agent 2 and the remaining 10% is split between all the agents

[21] As long as cross-monotonicity is satisfied.

in proportion to their power, and so forth. This rule satisfies generalized consistent ranking for the game $(N, \pi) = (\{1, 2, 3, 4, 5\}, [18.5, 21, 20, 19, 18.6])$ but does not satisfy consistent ranking. To see this, note that any coalition with three agents is contained in $SEC(N, \pi)$. However, the only coalition in the set $UD^{\xi}(N, \pi)$ is $\{1, 4, 5\}$ because this coalition is preferred by agents 1, 4 and 5 over any other coalition that contains them. Clearly consistent ranking is not satisfied, for instance, for the games with coalitions $\{1, 2, 3\}$ and $\{2, 3, 4\}$, agent 3 prefers the former whereas agent 2 prefers the latter.

The following result provides the complete class of sharing rules that allow the compatibility of SE and RAT. It also provides the unique transition correspondence that meets SE and RAT. Proposition 1(*ii*-*iii*) is a straighforward consequence of this result.

Proposition 2 *Consider the domain of games* $\boldsymbol{G}$*. There exists a transition correspondence* ϕ *defined on* $\boldsymbol{G}$ *that satisfies SE and RAT under the sharing rule* ξ *if and only if* ξ *satisfies GC. Moreover, if* ϕ *satisfies SE and RAT, then*

$$\phi(N, \pi) = \{T | T \in UD^{\xi}(N, \pi)\}.$$

4.3 Results under Combination Sharing

4.3.1 Size-Power Monotonic Games

In Example 1, we have demonstrated that combination sharing may not satisfy generalized consistent ranking. There is a class of games, however, where combination sharing will yield consistent ranking for any value λ (and hence satisfies GC for the restriction to this games). This class of games satisfy the condition of size-power monotonicity where larger-sized coalitions have more power.

Definition 8 (Size-Power Monotonicity (SPM)) *A game* (N, π) *is* ***size-power monotonic (SPM)*** *if for any* $A, B \subset N$ *such that* $|B| > |A|$*, we have that* $\pi(B) > \pi(A)$*. The set of SPM games is denoted by* $\bar{G}$*.*

What the SPM condition does is to take away the tension between coalition size and power (since power increases with size) and thus coalitions with smaller sizes (which, by definition, have lower power) will always give a higher share for any value of λ.

We say the domain of games $G \subset \mathbf{G}$ is feasible if for all $(N, \pi) \in G$, and for any $S \subset N$, the game $(S, \pi_S) \in G$. Proposition 3 characterizes the transition correspondence that satisfy SE and RAT over the domain of SPM games.

Proposition 3 *Let the transition correspondence ϕ^{**} be defined over the set $\bar{G}$ as follows:*

$$\phi^{**}(S,\pi) = \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M, \pi_M)$$

where $Q(S,\pi) = \{T \subset S \mid T \in W_{(S,\pi)}, T \in \phi^{*}(T,\pi_T)\}$

Under combination sharing CS^{λ},

i. for any fixed $\lambda \in (0,1)$, the transition correspondence ϕ defined on the domain of SPM games $\bar{G}$ satisfies SE and RAT if and only if $\phi = \phi^{**}$.

ii. if the transition correspondence ϕ defined on the feasible domain of games G satisfies SE and RAT for all combination sharing parameter $\lambda \in (0,1)$, then $\phi = \phi^{**}$ and $G \subseteq \bar{G}$.

The intuition behind the first part of Proposition 3 is that since SPM games guarantee that there is no disagreement with coalition size and power, then there will exist a ranking over these coalitions to which agents agree. Hence, combination sharing in this case satisfies consistent ranking and therefore we can use Proposition 1 to characterize the transition correspondence that will satisfy SE and RAT. The second part of Proposition 3 states that if we want to find a transition correspondence that satisfies SE and RAT for all possible values of $\lambda \in (0,1)$, then the domain of games should be a subset of $\bar{G}$, which is the set of SPM games.

However, there is a class beyond SPM games where a transition correspondence that satisfy SE and RAT exists for combination sharing provided that we restrict the parameter λ. What this restriction does is to make the set $UD^{CS^{\lambda}}(N,\pi)$ nonempty. Hence, combination sharing with the restricted λ values will satisfy generalized consistent ranking at these class of games.

4.3.2 Maximalist Domain

Consider the sharing rule CS^{λ}. Even if the game were not SPM (that is, there is a tension between coalition size and power), it is still possible to find conditions wherein a transition correspondence that satisfies SE and RAT will exist. In order to find such transition correspondence over these non-SPM games, we need to define the largest domain in which the parameter λ allows combination sharing to satisfy generalized consistent ranking.

In order to characterize the values of λ that will make combination sharing CS^{λ} satisfy generalized consistent ranking, we first note that from the definition of combination sharing that agents in the intersection of any two coalitions S and T will agree on coalition S if and only if their share is higher in S than in T, that is:

$$\lambda\frac{1}{|S|}+(1-\lambda)\frac{\pi_i}{\pi(S)}>\lambda\frac{1}{|T|}+(1-\lambda)\frac{\pi_i}{\pi(T)}\ \forall\ i\in S\cap T$$

or, after rearranging,

$$\lambda\frac{|T|-|S|}{|T||S|}+(1-\lambda)\pi_i\frac{\pi(T)-\pi(S)}{\pi(T)\pi(S)}>0\ \forall\ i\in S\cap T \tag{2}$$

For instance, if the size of coalition S were smaller than T ($|S|<|T|$) but the coalition power were higher ($\pi(S)>\pi(T)$), the the only way that S will be preferred is when the first term on the left-hand side of Equation 2 is higher than the second term.

Since all the parameters (coalition sizes, coalition powers, and the agents' power) in Equation 2 are known, in principle we can find values of λ where we can make all agents in the intersection of two coalitions prefer one over the other. That is, we can find a λ high enough for agents to prefer the smaller-sized coalition (since higher λ puts more weight towards equal sharing) and a λ low enough for the agents to prefer lower-powered coalitions (where proportional sharing dominates). Thus, for any two coalitions $S,T\subset 2^N$ we can define:

$$\underline{\lambda}^{(S,T)}(N,\pi)=\frac{\underline{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)}{\underline{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)+\left(\frac{|T|-|S|}{|T||S|}\right)} \tag{3}$$

where $\underline{\pi}_i^{(S,T)}$ is the power of agent i in the intersection S and T with the lowest power.

and

$$\bar{\lambda}^{(S,T)}(N,\pi)=\frac{\bar{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)}{\bar{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)+\left(\frac{|T|-|S|}{|T||S|}\right)} \tag{4}$$

where $\bar{\pi}_i^{(S,T)}$ is the power of agent i in the intersection of S and T with the highest power. Because of our assumption of no ties in the power of any pair of coalitions, these values of λ always exists.

In Equation 3, any $\lambda \leq \underline{\lambda}$ will convince agents in the intersection to choose S over T when S has the lower power by putting more weight on proportional sharing. Note that we only need to convince the agent with the lowest power $\underline{\pi}_i$ since he will have the least to gain under proportional sharing. If this lowest-powered agent's share is higher under this level of λ, then any other agent in the intersection of S and T with a higher power will also have a higher share in coalition S. In the same manner, Equation 4 provides the incentives to choose a smaller-sized coalition over a larger coalition (but with a lower power) by putting more weight on equal sharing. Note that only the agent in the intersection with the largest power must be convinced to choose the smaller-sized coalition since he has the most to lose in moving to a level of λ that puts more weight on equal sharing.

As a corollary, λ values between $\underline{\lambda}^{(S,T)}(N,\pi)$ and $\bar{\lambda}^{(S,T)}(N,\pi)$ create disagreement between the agents in the intersection of S and T. Particularly, when $\lambda > \underline{\lambda}^{(S,T)}(N,\pi)$ the lowest powered agent in the intersection prefers the coalition with the smaller size (and higher power) while for $\lambda < \bar{\lambda}^{(S,T)}(N,\pi)$ the highest powered agent in the intersection prefers the lower powered coalition (with larger size).

We are now in the position to calculate for non-SPM games $(N,\pi) \notin \bar{G}$ two threshold values $\underline{\Lambda}(N,\pi)$ and $\bar{\Lambda}(N,\pi)$ over the set $SEC(N,\pi)$. Recall from Definition 6 that the set $SEC(N,\pi)$ contains winning and self-enforcing coalitions from the game (N,π). The threshold value $\underline{\Lambda}(N,\pi)$ ensures that for any $\lambda \leq \underline{\Lambda}(N,\pi)$ the lowest-powered coalition in $SEC(N,\pi)$ will be unanimously preferred by all of its members over all other coalitions in $SEC(N,\pi)$ to which they could possibly belong. On the other hand, for $\lambda \geq \bar{\Lambda}(N,\pi)$ the smallest-sized coalition in $SEC(N,\pi)$ will be unanimously preferred by all of its members over all other coalitions in $SEC(N,\pi)$ to which they could possibly belong. These values are:

$$\underline{\Lambda}(N,\pi) = \min \underline{\lambda}^{(S,T)}(N,\pi) \quad \text{s.t. } \underline{\lambda}^{(S,T)}(N,\pi) \in (0,1) \tag{5}$$

for all $S,T \in SEC(N,\pi)$

and

$$\bar{\Lambda}(N,\pi) = \max \bar{\lambda}^{(S,T)}(N,\pi) \tag{6}$$

$$\text{s.t. } \bar{\lambda}^{(S,T)}(N,\pi) \in (0,1)$$

for all $S, T \in SEC(N,\pi)$

To illustrate these threshold values, consider the game

$$(N,\pi) = (\{1,2,3,4,5,6\}, [29, 26.5, 21, 2.49, 2.4, 2.39])$$

used in Example 1 in Section 4.2. This game is clearly not size-power monotonic since a two-person coalition like $\{4,5\}$ has less power than the singleton $\{1\}$.

In that game, we have the set

$$SEC(N,\pi) = \{\{1,2,3\}, \{1,2,5,6\}, \{1,2,4,5\}, \{1,2,4,6\}, \{2,3,4,5,6\}\}.$$

We calculate the following values based on Equations 3, 4, 5 and 6:

$\underline{\lambda}^{(\{1,2,3\},\{1,2,5,6\})}(N,\pi) = 0.5277$; $\bar{\lambda}^{(\{1,2,3\},\{1,2,5,6\})}(N,\pi) = 0.5501$

$\underline{\lambda}^{(\{1,2,3\},\{1,2,4,5\})}(N,\pi) = 0.5258$; $\bar{\lambda}^{(\{1,2,3\},\{1,2,4,5\})}(N,\pi) = 0.5482$

$\underline{\lambda}^{(\{1,2,3\},\{1,2,4,6\})}(N,\pi) = 0.5260$; $\bar{\lambda}^{(\{1,2,3\},\{1,2,4,6\})}(N,\pi) = 0.5484$

$\underline{\lambda}^{(\{1,2,3\},\{2,3,4,5,6\})}(N,\pi) = 0.4494$; $\bar{\lambda}^{(\{1,2,3\},\{2,3,4,5,6\})}(N,\pi) = 0.5074$

$\underline{\lambda}^{(\{1,2,5,6\},\{1,2,4,5\})}(N,\pi) = 1$; $\bar{\lambda}^{(\{1,2,5,6\},\{1,2,4,5\})}(N,\pi) = 1$

$\underline{\lambda}^{(\{1,2,5,6\},\{1,2,4,6\})}(N,\pi) = 1$; $\bar{\lambda}^{(\{1,2,5,6\},\{1,2,4,6\})}(N,\pi) = 1$

$\underline{\lambda}^{(\{1,2,5,6\},\{2,3,4,5,6\})}(N,\pi) = 0.0738$; $\bar{\lambda}^{(\{1,2,5,6\},\{2,3,4,5,6\})}(N,\pi) = 0.4692$

$\underline{\lambda}^{(\{1,2,4,5\},\{1,2,4,6\})}(N,\pi) = 1$; $\bar{\lambda}^{(\{1,2,4,5\},\{1,2,4,6\})}(N,\pi) = 1$

$\underline{\lambda}^{(\{1,2,4,5\},\{2,3,4,5,6\})}(N,\pi) = 0.0752$; $\bar{\lambda}^{(\{1,2,4,5\},\{2,3,4,5,6\})}(N,\pi) = 0.4733$

$\underline{\lambda}^{(\{1,2,4,6\},\{2,3,4,5,6\})}(N,\pi) = 0.0748$; $\bar{\lambda}^{(\{1,2,4,6\},\{2,3,4,5,6\})}(N,\pi) = 0.4729$

$\underline{\Lambda}(N,\pi) = 0.0738$

$\bar{\Lambda}(N,\pi) = 0.5501$

If $\lambda \leq \underline{\Lambda}(N,\pi)$, we are assured that agents in $\{2,3,4,5,6\}$ prefer this coalition to any other coalition in $SEC(N,\pi)$ where they could possibly belong. In this case, λ is "low enough" to encourage members to prefer the coalition with the least power. In this case, the set $UD^{CS^{\lambda}}(N,\pi)$ contains the coalition $\{2,3,4,5,6\}$ and is therefore non-empty. On the other hand, if $\lambda \geq \bar{\Lambda}(N,\pi)$, agents in the coalition $\{1,2,3\}$ prefer this coalition over any other coalition in $SEC(N,\pi)$ where they could possibly belong. In other words, λ is "high enough" to encourage the members to prefer the coalition with the smallest size. Here, the set $UD^{CS^{\lambda}}(N,\pi)$ contains the coalition $\{1,2,3\}$ and is likewise non-empty. For these extreme values of λ, combination sharing at the game $(N,\pi) = (\{1,2,3,4,5,6\},[29,26.5,21,2.49,2.4,2.39])$ satisfies generalized consistent ranking.

For this particular game, these extreme values of λ are not the only values where the set $UD^{CS^{\lambda}}(N,\pi)$ will be nonempty, as demonstrated in Example 1. In some games we may also find an interval within $\lambda \in (0,1)$ where a "compromise coalition" (that is, coalitions that neither have the least power nor the smallest size) will be unanimously preferred by its members. These compromise coalitions will exist whenever there exists a coalition Q where $|Q| \neq \min\limits_{T \in SEC(N,\pi)} |T|$ and $\pi(Q) \neq \min\limits_{T \in SEC(N,\pi)} \pi(T)$ such that:

$$\underline{\Lambda}^{Q}(N,\pi) = \min_{T \in SEC(N,\pi)} \bar{\lambda}^{(T,Q)}(N,\pi) < \max_{T \in SEC(N,\pi)} \underline{\lambda}^{(T,Q)}(N,\pi) = \bar{\Lambda}^{Q}(N,\pi) \tag{7}$$

subject to $\bar{\lambda}^{(T,Q)}(N,\pi) \in (0,1)$ and $\bar{\lambda}^{(T,Q)}(N,\pi) \in (0,1)\ \forall\ T,Q \in SEC(N,\pi)$

Whenever $\lambda \in \left[\underline{\Lambda}^{Q}(N,\pi), \bar{\Lambda}^{Q}(N,\pi)\right]$, the coalition $Q \in UD^{CS^{\lambda}}(N,\pi)$ and therefore this coalition is unanimously preferred by its intersecting members against any other coalition in $SEC(N,\pi)$.

In our current example, we have $\underline{\Lambda}^{Q}(N,\pi) = 0.4692$ and $\bar{\Lambda}^{Q}(N,\pi) = 0.5277$. Since this satisfies the condition in Equation 7, we are guaranteed that a compromise coalition exists and is unanimously preferred by its intersecting members against any other coalition in $SEC(N,\pi)$. Suppose that $\lambda = 0.5$ (which is contained in the interval $\left[\underline{\Lambda}^{Q}(N,\pi), \bar{\Lambda}^{Q}(N,\pi)\right]$), then the coalition $\{1,2,5,6\}$ will be unanimously preferred by its intersecting members. For instance, comparing $\{1,2,5,6\}$ and $\{1,2,3\}$, we have

$$\xi_1(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.3655$$

$$\xi_2(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.3447$$

$$\xi_5(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.1449$$

$$\xi_6(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.1448$$

$$\xi_1(\{1,2,3\},[29,26.5,21]) = 0.3562$$

$$\xi_2(\{1,2,3\},[29,26.5,21]) = 0.3398$$

$$\xi_5(\{1,2,3\},[29,26.5,21]) = 0$$

$$\xi_6(\{1,2,3\},[29,26.5,21]) = 0$$

It is easy to verify that the agents in $\{1,2,5,6\}$ will have the highest share in this coalition compared to any other coalition in $SEC(N,\pi)$, namely $\{1,2,3\}$, $\{1,2,4,5\}$, $\{1,2,4,6\}$ and $\{2,3,4,5,6\}$. Hence, $\{1,2,5,6\} \in UD^{CS^{0.5}}(N,\pi)$.

Of course there are games where we cannot find these compromise coalitions. For instance, suppose we decrease agent 3's power to 20 instead of 21 so that the game now becomes:

$$(N,\tilde{\pi}) = (\{1,2,3,4,5,6\},[29,26.5,20,2.49,2.4,2.39]).$$

In this example, the compromise coalition $\{1,2,5,6\}$ is not part of $UD^{CS^{0.5}}(N,\tilde{\pi})$ because Equation 7 is not satisfied, that is,

$$\underline{\Lambda}^Q(N,\tilde{\pi}) = 0.5155 \not< \bar{\Lambda}^Q(N,\tilde{\pi}) = 0.5151$$

The next proposition establishes the largest class of non-SPM games for which there exists a transition correspondence satisfying SE and RAT. In these games we basically need the combination sharing parameter λ to be either low enough for agents to unanimously pick the coalition with the least power, high enough for them to pick the coalition with the smallest size, or inside the interval where they pick the compromise coalition.

Formally, consider the classes of games such that

$$G^*(\lambda) = \{(S,\pi_S) | \lambda \in (0,\underline{\Lambda}(S,\pi_S)] \cup [\bar{\Lambda}(S,\pi_S),1) \cup \bigcup_{Q \in SEC(S,\pi_S)} [\bar{\Lambda}^Q(S,\pi_S),\underline{\Lambda}^Q(S,\pi_S)]\}$$

Proposition 4 *Consider the sharing rule CS^λ for a fixed $\lambda \in (0,1)$ and let $\breve{\phi}$ be a transition correspondence defined over the feasible set of games $G \subseteq \boldsymbol{G}$. Then,*

a. *CS^λ satisfies GC at $(N,\pi) \in G$ if and only if $G \subseteq G^*(\lambda)$*

b. For the transition correspondence $\breve{\phi}: G^*(\lambda) \to 2^N$ that satisfies SE and RAT:

i. if $\lambda \in \big[\bar{\Lambda}(N,\pi), 1\big)$,

$$\breve{\phi}(N,\pi) = \underset{T \in SEC(N,\pi)}{\arg\min} |T|$$

ii. if $\lambda \in (0, \underline{\Lambda}(N,\pi)]$,

$$\breve{\phi}(N,\pi) = \underset{T \in SEC(N,\pi)}{\arg\min} \pi(T)$$

iii. if $\underline{\Lambda}^Q(N,\pi) \leq \lambda \leq \bar{\Lambda}^Q(N,\pi)$

$$\breve{\phi}(N,\pi) = \big\{Q \mid Q \in SEC(N,\pi), \underline{\Lambda}^Q(N,\pi) < \bar{\Lambda}^Q(N,\pi)\big\}$$

Remark 1 *i. For any coalition formation game (N,π), there is always a combination sharing parameter λ such that $(N,\pi) \in G^*(\lambda)$ and therefore CS^λ at this game (N,π) satisfies generalized consistent ranking. Thus, at such λ there exists a transition correspondence $\breve{\phi}$ that satisfies SE and RAT at the game (N,π).*

ii. For any coalition formation game (N,π) and sharing rule CS^λ, there is always a coalition $S \subset N$ who can give up their power to $\tilde{\pi}_S \leq \pi_S$ such that the game $(N,(\tilde{\pi}_S,\pi_{-S})) \in G^*(\lambda)$. Thus, at this new power profile there exists a transition correspondence $\breve{\phi}$ that satisfies SE and RAT at the game $(N,(\tilde{\pi}_S,\pi_{-S})) \in G^*(\lambda)$.

Hence, a lesson from the remark is that potential disagreements among agents on the choice of coalitions need not lead to a complete breakdown of cooperation. Regardless of the initial configuration of power, it is always possible to find agreements on coalitions if only agents agree to either set the correct method of sharing the prize (λ) or, alternatively, to give up power.[22] This is an optimistic observation, and can very well be applied to any activity that requires cooperation.

5 Conclusion

This paper develops an axiomatic approach to a coalition formation model by focusing on two main axioms: self-enforcement and rationality. We investigate the effect

[22]Giving up power is just one way to change the power profile. Another way of changing such a profile includes the transferring of power between agents. An open question that we leave for future studies is to find the 'minimal conditions' on the change of the power to guarantee the stability of the sharing rule CS^λ.

of different sharing rules on the existence of transition correspondences that satisfy these axioms. Our results show that the existence of these transition correspondences is very sensitive to the choice of sharing rules.

We find that when the sharing rule satisfies the property of consistent ranking (where agents have the same ordinal rank over coalitions) then we can always find a transition correspondence that satisfies these axioms. Moreover, this result extends to sharing rules that satisfy generalized consistent ranking where coalition members unanimously agree to prefer one coalition over all other self-enforcing and winning coalitions.

Under combination sharing, however, these transition correspondences do not exist in general. In order for combination sharing to satisfy generalized consistent ranking, we have to restrict the domain of games either to the case where coalition size and power move in the same direction or by allowing the sharing parameter λ to be high enough for agents to agree on the smallest-sized undominated coalition, low enough for agents to agree on the least-powered undominated coalition, or just enough for a compromise coalition to exist.

A Appendix: Proofs

Proof of Lemma 1

Proof. Consider the sets

$$A^u = \{S | S \in \phi(S, \pi), |S| \leq u\}$$

and

$$B^u = \{T | T \in \tilde{\phi}(T, \pi), |T| \leq u\}.$$

We will prove by induction on the size of u that $A^u = B^u$.

This is clearly true if $u = 1$, because any singleton coalition is self-enforcing.

For the induction hypothesis, assume that $A^{u-1} = B^{u-1}$.

Consider $S \in A^u$. Then $S \in \phi(S, \pi)$. Therefore, since ϕ is minimalistic, there is no $Q \subset S$ such that $Q \in W_{(S,\pi)}$ and $Q \in \phi(Q, \pi_Q)$.

Therefore, since $A^{u-1} = B^{u-1}$, there is no $Q \subset S$ such that $Q \in W_{(S,\pi)}$ and $Q \in \tilde{\phi}(Q, \pi_Q)$.

Hence, $S \in \tilde{\phi}(S, \pi)$ and $S \in B^u$. Thus $A^u \subset B^u$.

We can similarly prove that $B^u \subset A^u$. ■

Proof of Proposition 1

Proof.

Step 1. ϕ^* is SE and minimalistic.

Proof. To show SE, take any $X \in \phi^*(S,\pi)$. There are two cases: either $X = S$ or $X \in Q(S,\pi)$. If $X = S$, then $X \in \phi^*(S,\pi) = \phi^*(X,\pi_X)$. If $X \in Q(S,\pi)$, then $X \in \phi^*(X,\pi_X)$ by definition of the set $Q(S,\pi)$.

On the other hand, ϕ^* is minimalistic because ξ is cross-monotonic. That is, at the coalition formation game (S,π), the set S is chosen only if $Q(S,\pi) = \emptyset$.

Step 2. ϕ^* satisfies RAT.

Proof. Take $T \in \phi^*(S,\pi)$ and consider a coalition Z such that $Z \in W_{(S,\pi)}$ such that $Z \in \phi^*(Z,\pi_Z)$.

($\Rightarrow$) First assume that $Z \notin \phi^*(S,\pi)$. Since $T \in \phi^*(S,\pi)$ we have that

$$T \in \underset{M \in Q(S,\pi)\cup\{S\}}{\arg\max} R^{\xi}(M,\pi_M)$$

Notice that Z is winning and self-enforcing within S, therefore $Z \in Q(S,\pi) \cup \{S\}$. Moreover, since $Z \notin \phi^*(S,\pi)$, then $Z \notin \underset{M \in Q(S,\pi)\cup\{S\}}{\arg\max} R^{\xi}(M,\pi_M)$. Hence, $R^{\xi}(T,\pi_T) > R^{\xi}(Z,\pi_Z)$.

($\Leftarrow$) Now, assume that $R^{\xi}(T,\pi_T) > R^{\xi}(Z,\pi_Z)$. Then, $Z \notin \underset{M \in Q(S,\pi)\cup\{S\}}{\arg\max} R^{\xi}(M,\pi_M)$. Hence, $Z \notin \phi^*(S,\pi)$

Step 3. Consider any cross-monotonic sharing rule and transition correspondences ϕ and $\tilde{\phi}$ that are self-enforcing and minimalistic. Then, the sets of coalitions that are self-enforcing coincide. That is,

$$\{S | S \in \phi(S,\pi)\} = \{T | T \in \tilde{\phi}(T,\pi)\}.$$

Proof. The proof of this result is a straightforward consequence of Lemma 1.

Step 4. There exists a unique transition correspondence that meets SE and RAT.

Proof. Consider a transition correspondence ϕ that is SE and RAT. Then, ϕ is minimalistic because the sharing rule is cross-monotonic. We will show that $\phi = \phi^*$.

Since ϕ and ϕ^* are SE and RAT, then by step 3,

$$\{T | T \in \phi(T,\pi_T)\} = \{T | T \in \phi^*(T,\pi_T)\} \tag{8}$$

Suppose $X \in \phi(X, \pi)$. Then, by equation 8, $X \in \phi^*(X, \pi)$. By RAT, for any $S \subset X$, $S \neq X$, then $S \notin \phi(X, \pi)$. Hence, $\phi(X, \pi) = \phi^*(X, \pi)$.

On the other hand, suppose $S \in \phi(X, \pi)$, where $S \neq X$. Then, by RAT, $\xi_i(S, \pi_S) \geq \xi_i(T, \pi_T)$ for $i \in S \cap T$ for any coalition T such that $T \in \phi(T, \pi_T)$ and $T \in W_{(X,\pi)}$. Therefore, by consistent ranking, $R^\xi(S, \pi_S) \geq R^\xi(T, \pi_T)$ for any coalition T such that $T \in \phi(T, \pi_T)$ and $T \in W_{(X,\pi)}$. Hence, $S \in \phi^*(X, \pi)$ and $\phi(X, \pi) \subset \phi^*(X, \pi)$. We could similarly show that $\phi^*(X, \pi) \subset \phi(X, \pi)$. Therefore, $\phi(X, \pi) = \phi^*(X, \pi)$.

Step 5. ϕ^* is superior to any transition correspondence that is SE and minimalistic.

Proof. We prove this step by contradiction. Suppose ϕ^* is not superior to the SE and minimalistic transition correspondence $\hat{\phi}$. Then, there exists a game (N, π) such that $S, T \subset N$ where $S \in \phi^*(N, \pi)$ and $T \in \hat{\phi}(N, \pi)$ such that $\xi_i(T, \pi_T) \geq \xi_i(S, \pi_S)$ for some $i \in T \cap S$.

By step 3, since T is self-enforcing for $\hat{\phi}$, then it is also self-enforcing for ϕ^*. Therefore, $T \in Q(N, \pi)$.

Since $T \notin \phi^*(N, \pi)$, then $T \notin \underset{M \in Q(N,\pi) \cup \{N\}}{\arg\max} R^\xi(M, \pi_M)$. Since $S \in \phi^*(N, \pi)$, then $S \in \underset{M \in Q(N,\pi) \cup \{N\}}{\arg\max} R^\xi(M, \pi_M)$. Since S and T are winning within N (by the definition of a transition correspondence), we have that $T \cap S \neq \emptyset$. Therefore, $R^\xi(S, \pi_S) > R^\xi(T, \pi_T)$, which implies that $\xi_i(S, \pi_S) > \xi_i(T, \pi_T)$ for $i \in T \cap S$. This is a contradiction.

■

Proof of Proposition 2

Proof.

We prove this result in four steps.

Step 1. Suppose ϕ and $\tilde{\phi}$ are transition correspondences that satisfy SE and RAT under the sharing rule ξ. Then, $\phi = \tilde{\phi}$.

Proof. Suppose there is a game (N, π) such that $\phi(N, \pi) \neq \tilde{\phi}(N, \pi)$. Without loss of generality, let $T \in \tilde{\phi}(N, \pi) \setminus \phi(N, \pi)$ and let $S \in \phi(N, \pi)$. Note that S and T are self-enforcing under ϕ and $\tilde{\phi}$ by Lemma 1. Hence, by the rationality of ϕ, $\xi_i(S, \pi_S) > \xi_i(T, \pi_T)$ for all $i \in T \cap S$.

If $S \in \tilde{\phi}(N, \pi)$, then by the rationality of $\tilde{\phi}$, $T \notin \tilde{\phi}(N, \pi)$. This is a contradiction. On the other hand, if $S \notin \tilde{\phi}(N, \pi)$, then by the rationality of $\tilde{\phi}$, we have that $\xi_i(S, \pi_S) < \xi_i(T, \pi_T)$ for all $i \in T \cap S$. This is a contradiction.

Therefore, $\tilde{\phi}(N, \pi) \setminus \phi(N, \pi) = \emptyset$. Hence, $\tilde{\phi}(N, \pi) \subset \phi(N, \pi)$. By a similar argument we can show that $\phi(N, \pi) \subset \tilde{\phi}(N, \pi)$.

Step 2. Let

$$DOM(N,\pi) = SEC(N,\pi) \setminus UD^{\xi}(N,\pi)$$

be the set of dominated coalitions in (N,π). Then, for any ϕ that satisfies RAT and SE, we have that

$$\phi(N,\pi) \cap DOM(N,\pi) = \emptyset.$$

Proof. We prove it by contradiction. Suppose that $S \in \phi(N,\pi) \cap DOM(N,\pi)$. Then, there exists $T \in SEC(N,\pi)$ such that $\xi_i(T,\pi_T) > \xi_i(S,\pi_S)$ for all $i \in S \cap T$. Since ϕ satisfies rationality, then it also satisfies minimalistic. Therefore, by Lemma 1, $T \in \phi(T,\pi_T)$.

If $T \in \phi(N,\pi)$, then $S \notin \phi(N,\pi)$ by rationality, which is a contradiction.

On the other hand, if $T \notin \phi(N,\pi)$, then $\xi_i(S,\pi_S) > \xi_i(T,\pi_T)$ for all $i \in S \cap T$. This is a contradiction.

Hence, $\phi(N,\pi) \cap DOM(N,\pi) = \emptyset$.

Step 3. We prove necessity.

(i.) To show that the set $UD^{\xi}(N,\pi)$ is nonempty.

Consider the transition correspondence ϕ that satisfies SE and RAT. First, note that RAT implies MIN; therefore, ϕ satisfies the conditions of Lemma 1. Let (N,π) be a coalition formation game and let $S \in \phi(N,\pi)$. By SE, $S \in \phi(S,\pi_S)$. Therefore, $S \in SEC(N,\pi)$.

By step 2, $S \in UD^{\xi}(N,\pi)$. Therefore, $UD^{\xi}(N,\pi) \neq \emptyset$.

(ii.) To show that $\phi(N,\pi) = \{T | T \in UD^{\xi}(N,\pi)\}$.

(a.) Let $T \in UD^{\xi}(N,\pi)$. Then, $T \in SEC(N,\pi)$ and therefore $T \in \phi(T,\pi_T)$.

If $T \notin \phi(N,\pi)$, then there exists another coalition $S \in SEC(N,\pi)$ such that $\xi_i(S,\pi_S) > \xi_i(T,\pi_T)$ for all $i \in T \cap S$. Thus, $T \notin UD^{\xi}(N,\pi)$. This is a contradiction. Therefore, $T \in \phi(N,\pi)$. Hence,

$$\{T | T \in UD^{\xi}(N,\pi)\} \subset \phi(N,\pi).$$

(b.) Let $T \in \phi(N,\pi)$. Then, T is self-enforcing. Thus, $T \in SEC(N,\pi)$. By step 2, $T \notin DOM(N,\pi)$. Hence, $T \in UD^{\xi}(N,\pi)$. Therefore,

$$\phi(N,\pi) \subset \{T | T \in UD^{\xi}(N,\pi)\}.$$

Hence,

$$\phi(N,\pi) = \{T|T \in UD^{\xi}(N,\pi)\}.$$

Step 4. We prove sufficiency by constructing a transition correspondence $\hat{\phi}$ and showing that it satisfies SE and RAT. Note that, by Step 1, this transition correspondence $\hat{\phi}$ will be unique.

Assume that $\hat{\phi} = \{T|T \in UD^{\xi}(N,\pi)\}$.

(i.) To show that $\hat{\phi}$ satisfies SE.

If $S \in \hat{\phi}(N,\pi)$, then $S \in UD^{\xi}(N,\pi)$. By Definition 6, $S \in SEC(N,\pi)$

(ii.) To show that $\hat{\phi}$ satisfies RAT.

Suppose that $S \notin \hat{\phi}(N,\pi)$ where $S \in SEC(N,\pi)$.

Let coalition $T \in \hat{\phi}(N,\pi)$. Hence, by definition of $\hat{\phi}$, we have $T \in UD^{\xi}(N,\pi)$. By definition of $UD^{\xi}(N,\pi)$, we have that

$$\xi_i(T,\pi_T) > \xi_i(S,\pi_S)$$

for all $S \in SEC(N,\pi)$ and for all $i \in S \cap T$.

■

Proof of Proposition 3

Proof.

To prove part (i.), let the game be $(N,\pi) \in \bar{G}$ and assume combination sharing and fix a λ.

For any game $(S,\pi_S) \subset (N,\pi)$, agent i's share in a coalition S is:

$$\xi_i(S,\pi_S) = \lambda \cdot \frac{1}{|S|} + (1-\lambda) \cdot \frac{\pi_i}{\pi(S)}$$

Since $(N,\pi) \in \bar{G}$, If there exists another game $(T,\pi_T) \subset (N,\pi)$ such that $|S| < |T|$, it must also be true that $\pi(S) < \pi(T)$. Therefore,

$$\xi_i(S,\pi_S) > \xi_i(T,\pi_T)$$

for all agents in $i \in S \cap T$.

Hence, we can construct a ranking function

$$R^{CS^\lambda}(S,\pi) = \lambda \cdot \frac{1}{|S|} + (1-\lambda) \cdot \frac{\pi_i}{\pi(S)}$$

where coalitions with smaller size (and therefore lesser power by size-power monotonicity) will yield a higher value of $R^{CS^\lambda}(S,\pi)$. Hence, the sharing rule satisfies consistent ranking and by Proposition 1, the transition correspondence ϕ^{**} is the unique transition correspondence satisfying SE and RAT over the domain of SPM games.

To prove part (ii), assume a transition correspondence ϕ that satisfies SE and RAT. Suppose we have two arbitrary coalitions S and T, such that $(S,\pi_S) \in \bar{G}$ and $(T,\pi_T) \in \bar{G}$. With combination sharing, agents in the intersection of S and T will have a higher share in S whenever

$$\xi_i(S,\pi_S) = \lambda\frac{1}{|S|} + (1-\lambda)\frac{\pi_i}{\pi(S)} > \lambda\frac{1}{|T|} + (1-\lambda)\frac{\pi_i}{\pi(T)} = \xi_i(T,\pi_T)$$

As $\lambda \to 1$, $\xi_i(S,\pi_S) \to \frac{1}{|S|}$ and $\xi_i(T,\pi_T) \to \frac{1}{|T|}$. Hence to maintain the inequality, it must be true that $|S| < |T|$. As $\lambda \to 0$, $\xi_i(S,\pi_S) \to \frac{\pi_i}{\pi(S)}$ and $\xi_i(T,\pi_T) \to \frac{\pi_i}{\pi(T)}$. Hence to maintain the inequality, it must be true that $\pi(S) < \pi(T)$. Therefore, for a ranking function R^{CS^λ} to exist for all values of λ and for all games in $\bar{G}$, we should have that for any two coalitions S and T, such that $(S,\pi_S) \in \bar{G}$ and $(T,\pi_T) \in \bar{G}$, if $|S| < |T|$ then $\pi(S) < \pi(T)$.

The proof that $\phi = \phi^{**}$ is similar to Step 4 in the proof of Proposition 1, restricted to the domain of SPM games.

■

Proof of Proposition 4

Proof. To prove part (a) ($\Rightarrow$), suppose that CS^λ satisfies GC at (N,π). Then the set $UD^{CS^\lambda}(N,\pi) \neq \emptyset$ and there must be at least one coalition $S \in SEC(N,\pi)$ such that $\xi_i(S,\pi_S) > \xi_i(T,\pi_T)$ for all $i \in S$ and any other coalition $T \in SEC(N,\pi)$

Case 1:. $S = \underset{T \in SEC(N,\pi)}{\text{argmin}} |T|$

In this case, it must be true that the highest powered agent must be convinced to stay at S because he has the most to lose when moving into a higher powered coalition (with smaller size) and this happens when

$$\lambda > \max \bar{\lambda}^{(S,T)}(N,\pi) = \bar{\Lambda}(N,\pi)$$

$$\text{s.t. } \bar{\lambda}^{(S,T)}(N,\pi) \in (0,1)$$

This is the interval $[\bar{\Lambda}(N,\pi), 1)$

Case 2:. $S = \underset{T \in SEC(N,\pi)}{\text{argmin}} \; \pi(T)$

In this case, it must be true that the lowest powered agent must be convinced to stay at S because he has the most to lose when moving into a lower powered coalition and this happens when

$$\lambda < \min \underline{\lambda}^{(S,T)}(N,\pi) = \underline{\Lambda}(N,\pi)$$

$$\text{s.t. } \underline{\lambda}^{(S,T)}(N,\pi) \in (0,1)$$

This is the interval $(0, \underline{\Lambda}(N,\pi)]$

Case 3:. $S \neq \underset{T \in SEC(N,\pi)}{\text{argmin}} \; |T|$ nor $S \neq \underset{T \in SEC(N,\pi)}{\text{argmin}} \; \pi(T)$

In this case, the highest powered agent in S must be compensated by putting just enough weight on equal sharing to make him choose the smaller compromise coalition but not too much to incentivize him to join the coalition of the least power. This happens when

$$\lambda > \min \bar{\lambda}^{(S,T)}(N,\pi)$$

$$\text{s.t. } \bar{\lambda}^{(S,T)}(N,\pi) \in (0,1)$$

One also has to compensate the lowest powered agent just enough for him to choose the compromise coalition but not to incentivize him to choose the smallest sized coalition. This happens when

$$\lambda < \max \underline{\lambda}^{(S,T)}(N,\pi)$$

$$\text{s.t. } \underline{\lambda}^{(S,T)}(N,\pi) \in (0,1)$$

Taken together for all coalitions of this type, the relevant interval is

$$\bigcup_{Q \in SEC(N,\pi)} \left[\bar{\Lambda}^{Q}(N,\pi), \underline{\Lambda}^{Q}(N,\pi)\right]$$

The intervals in all these cases constitute the admissible values of λ for the game (N,π) to constitute the set $G^*(\lambda)$

($\Leftarrow$) Suppose $(N,\pi) \in G^*(\lambda)$. Then there exist the interval $(0, \underline{\Lambda}(N,\pi)]$ where $S = \underset{T \in SEC(N,\pi)}{\text{argmin}} \; \pi(T)$ is an element of $UD^{CS^{\lambda}}(N,\pi)$ since members of S unanimously pick S because this coalition gives them the highest share over all coalitions in $SEC(N,\pi)$at these values of λ. There also exists the interval $[\bar{\Lambda}(N,\pi),1)$ where $S = \underset{T\in SEC(N,\pi)}{\text{argmin}} |T|$ is an element of $UD^{CS^\lambda}(N,\pi)$ since members of S unanimously pick S because this coalition gives them the highest share over all coalitions in $SEC(N,\pi)$ at these values of λ. In some cases, there also exists an interval $\left[\bar{\Lambda}^Q(N,\pi),\underline{\Lambda}^Q(N,\pi)\right]$ for a coalition Q where $|Q| \neq \min_{T\in SEC(N,\pi)} |T|$ and $\pi(Q) \neq \min_{T\in SEC(N,\pi)} \pi(T)$. This coalition Q is an element of $UD^{CS^\lambda}(N,\pi)$ at this interval since members of Q unanimously pick Q because this coalition gives them the highest share over all coalitions in $SEC(N,\pi)$ at these values of λ.

Proof of part $(b.i)$

Suppose $\lambda \in \left[\bar{\Lambda}(N,\pi),1\right)$ and that $S \in \breve{\phi}(N,\pi)$. Assume that $S \neq \underset{T\in SEC(N,\pi)}{\arg\min} |T|$. Then for the coalition $M \in SEC(N,\pi)$ where $M = \underset{T\in SEC(N,\pi)}{\arg\min} |T|$ we have that for all the agents $i \in M\cap S$ we have that $\xi_i(M,\pi_M) > \xi_i(S,\pi_S)$. Hence, by RAT, $S \notin \breve{\phi}(N,\pi)$, a contradiction. Conversely, values of λ in this interval guarantees that the set $UD^{CS^\lambda}(N,\pi)$ is non-empty since there will always be a coalition that will yield a highest share for all its members in the set $SEC(N,\pi)$. Therefore, combination sharing CS^λ at the game (N,π) satisfies generalized consistent ranking. By Proposition 2, and the fact that transition correspondence $\breve{\phi}(N,\pi)$ picks the coalitions in the set $UD^{CS^\lambda}(N,\pi)$, means that $\breve{\phi}(N,\pi)$ satisfies SE and RAT.

Proof of part $(b.ii)$

Suppose $\lambda \in (0,\underline{\Lambda}(N,\pi)]$ and that $S \in \breve{\phi}(N,\pi)$. Assume that $S \neq \underset{T\in SEC(N,\pi)}{\arg\min} \pi(T)$. Then for the coalition $M \in SEC(N,\pi)$ where $M = \underset{T\in SEC(N,\pi)}{\arg\min} \pi(T)$ we have that for all the agents $i \in M\cap S$ we have that $\xi_i(M,\pi_M) > \xi_i(S,\pi_S)$. Hence, by RAT, $S \notin \breve{\phi}(N,\pi)$, a contradiction. Conversely, values of λ in this interval guarantees that the set $UD^{CS^\lambda}(N,\pi)$ is non-empty since there will always be a coalition that will yield a highest share for all its members in the set $SEC(N,\pi)$. Therefore, combination sharing CS^λ at the game (N,π) satisfies generalized consistent ranking. By Proposition 2, and the fact that transition correspondence $\breve{\phi}(N,\pi)$ picks the coalitions in the set $UD^{CS^\lambda}(N,\pi)$, means that $\breve{\phi}(N,\pi)$ satisfies SE and RAT.

Proof of part $(b.iii)$

Suppose $\underline{\Lambda}^Q(N,\pi) \leq \lambda \leq \bar{\Lambda}^Q(S,\pi)$ and that $S \in \breve{\phi}(N,\pi)$. Assume that $S \notin \left\{Q \mid Q \in SEC(N,\pi), \underline{\Lambda}^Q(N,\pi) < \bar{\Lambda}^Q(N,\pi)\right\}$. Then for the coalition $M \in SEC(N,\pi)$

where $M \in \{Q \mid Q \in SEC(N,\pi), \underline{\Lambda}^Q(S,\pi_S) < \bar{\Lambda}^Q(S,\pi_S)\}$ we have that for all the agents $i \in M \cap S$ we have that $\xi_i(M,\pi_M) > \xi_i(S,\pi_S)$. Hence, by RAT $S \notin \breve{\phi}(N,\pi)$, a contradiction. Conversely, values of λ in this interval guarantees the set $UD^{CS^\lambda}(N,\pi)$ is non-empty since there will always be a coalition that will yield a highest share for all its members in the set $SEC(N,\pi)$. Therefore, combination sharing CS^λ at the game (N,π) satisfies generalized consistent ranking. By Proposition 2, and the fact that transition correspondence $\breve{\phi}(N,\pi)$ picks the coalitions in the set $UD^{CS^\lambda}(N,\pi)$, means that $\breve{\phi}(N,\pi)$ satisfies SE and RAT.

■

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